

# An example elucidating the mathematical situation in the statistical non-uniqueness problem of turbulence

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## Abstract

An instructive example is presented to elucidate the mathematical situation in the non-uniqueness problem of the infinite Friedmann-Keller hierarchy of equations for all multi-point moments within the theory of spatially unbounded Navier-Stokes turbulence. It is shown that the non-uniqueness problem of the Friedmann-Keller hierarchy emerges from the property that the system of equations is defined forward recursively. As a result, this system does not possess a unique general solution, even when the complete infinite system is formally considered. That is, even when imposing a sufficient number of initial conditions to this infinite system, it still does not provide a unique solution. This finding is supported by a Lie-group invariance analysis, in that the imposed example analogous to the Friedmann-Keller hierarchy admits an unclosed Lie algebra which allows for infinitely many functionally different equivalence transformations which all can be made compatible with any specifically chosen initial condition. Hence, if no prior modelling assumptions are made to close the Friedmann-Keller system of equations, the existence of an invariant solution within such a forward recursively defined system is then without value, since it just represents an arbitrary solution among infinitely many other, equally privileged invariant solutions.

**Keywords:** *Turbulence, Hopf Equation, Friedmann-Keller Hierarchy, Multi-point Correlations, Functional Equations, Ordinary and Partial Differential Equations, Infinite Systems, Lie Groups, Lie Algebra, Symmetries and Equivalences, Unclosed Systems, General Solutions, Cauchy Problems*

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## 1. Introduction and motivation

Due to the non-linear evolution of the Navier-Stokes equations, the statistical description of a turbulent flow on the level of the velocity moments is inherently unclosed. Presumably the first attempt to systematically formulate the equations of moments for turbulence was given by the proposed method of Keller & Friedmann (1924)<sup>†</sup>. The result is an infinite hierarchy of coupled multi-point correlation (MPC) equations. Theoretical considerations show that on the level of the moments all (infinite) correlation orders have to be taken into account in order to allow for a consistent statistical description of turbulence, which, since then, became known as the still prevailing closure problem of turbulence. The motivation of Hopf (1952) was to formally bypass this problem in formulating a statistical description which operates on a single closed equation: Known as the functional Hopf equation<sup>‡</sup>, it then induces the unclosed infinite hierarchy of MPC equations through a single moment generating functional  $\Phi$ .

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<sup>†</sup>For a more concise derivation of the Friedmann-Keller equations, see e.g. Monin & Yaglom (1971).

<sup>‡</sup>Note that also the discrete version of the Hopf equation, the Lundgren-Monin-Novikov hierarchy, which constitutes an infinite chain of coupled integro-differential equations, remains to be a closed system. The reason is that next to any possible initial conditions sufficient internal integral conditions are defined such that its general solution manifold is unique. For more details, see e.g. Hosokawa (2006) along with Appendix C in Frewer *et al.* (2014a) and the references therein.

Although one gained through this extension the theoretical advantage of dealing with a formally closed statistical description, the practical situation of constructing specific solutions of the problem, however, did not improve substantially when compared to the unclosed description of the moments. The fact is that on both levels of statistical description we face serious drawbacks: On the higher, more abstract level of the Hopf equation we face the problem of dealing with a functional calculus which from the outset is difficult to access numerically as well as analytically in a practical and constructive sense, while on the lower level of the Friedmann-Keller equations we face the problem of dealing with a hierarchy of coupled partial differential equations (PDEs) which is infinite. As I will show in this study by analyzing a lower-dimensional analogue, it becomes apparent that the latter problem faces a yet more fundamental problem: Even when formally considering all (infinite) equations in the statistical hierarchy, i.e. even if the Friedmann-Keller system is *not* truncated, the full system itself is still unclosed since no unique *general* solution can be constructed. In other words, when considering the case of a spatially unbounded flow e.g. in stating it mathematically as a Cauchy problem (as will be done throughout this study), the infinite system still is underdetermined and thus unclosed in that it does not offer any unique solutions although specific initial conditions have been posed. The reason for this defect is discussed in this study at the example of a lower-dimensional analogue, which shows all primary features of the Hopf-induced Friedmann-Keller hierarchy of MPC equations when stated as a functional Cauchy problem. These features are:

- i) the linearity of the higher level equation (the Hopf equation is a linear equation),
- ii) the linearity of the induced lower level moment equations (the infinite Friedmann-Keller hierarchy is a linear system),
- iii) the infinite hierarchy of moments is ordered forward recursively (each equation in the Friedmann-Keller hierarchy contains a higher order correlation function which only enters the next higher order equation in this chain).

The non-uniqueness problem just mentioned above is rooted solely in feature iii), in that a certain hierarchy of differential equations is defined *forward* recursively. Because, as I will demonstrate, if the infinite chain of equations would be ordered oppositely, namely *backward* recursively (where each equation contains a lower order correlation function which then only enters the next lower order equation in this chain), then the non-uniqueness problem does not exist. But since the infinite Friedmann-Keller chain of moments is defined as a forward recursive system, it inherently faces the problem of non-uniqueness in its *general* solution manifold even if all infinite equations of the system are formally taken along. Consequently, when performing an invariance analysis on such a system, one can generate infinitely many and functionally diverse Lie-point symmetries<sup>†</sup>, and hence infinitely many different and independent invariant solutions which all can be made compatible to any posed initial condition.

My motivation for this study was to elucidate this non-uniqueness problem for the MPC equations in all its facets, because it seems that in the relevant literature on turbulence there still exists a misconception on this issue, in particular in the studies of Oberlack et al., e.g. in Oberlack & Rosteck (2010), Oberlack & Zieleniewicz (2013), Avsarkisov *et al.* (2014), Waclawczyk *et al.* (2014), and Oberlack *et al.* (2014). In all these studies the reduced (lower level) Friedmann-Keller hierarchy of multi-point moments is incorrectly treated as a *closed*

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<sup>†</sup>In order to warrant consistency within an invariance analysis as for the Friedmann-Keller system, the actual symmetries must be identified as weaker equivalences. The reason is that since the considered system of equations is unclosed, all admitted invariances only act as equivalence transformations and not as symmetry transformations. For more details, see e.g. Section 2 in Frewer *et al.* (2014a), Sections 3-5 in Frewer *et al.* (2014b), and the references therein.

system, with the misleading argument that it's due to that the system is infinite dimensional, because "... if the infinite set of correlation equations is considered the closure problem is somewhat bypassed" (Oberlack & Rosteck, 2010, p. 454). Also the performed invariance analysis in each of these studies is misleading. For example, to obtain the result that "... the latter set of symmetries is considerably enlarged for the infinite set of MPC equations" (Oberlack & Zieleńiewicz, 2013, p. 12)<sup>†</sup> is not a surprising result, because the considered infinite Friedmann-Keller system of moments, as we will assert herein due to its forward recursively organized hierarchy, is simply an unclosed and thus an underdetermined system of equations which intrinsically allows for arbitrary invariances. Furthermore, to address the open question that "... so far completeness of all admitted symmetries of the MPC equation has not been shown... [which] appears to be necessary not only from a theoretical point of view but rather essential to generate scaling for all higher moments" (Oberlack *et al.*, 2014, p. 1702), and that "... hence, finding new statistical symmetries and/or proving the completeness of the set of symmetries is a next task for further study" (Wacławczyk *et al.*, 2014, p. 10), would collectively only lead to a superfluous task, because, as already said, arbitrary invariances can be constructed when considering an unclosed infinite hierarchy of equations as the MPC equations considered by Oberlack *et al.*; therefore completeness in this very sense can never be obtained in such infinite systems. Finally, when trying to correctly interpret any results arising from such an invariance analysis, in particular when trying to construct invariant solutions for such unclosed systems, it is necessary to recognize the subtle but important difference between a symmetry transformation and an equivalence transformation. While a symmetry transformation always maps a solution to another solution of the same (closed) equation, an equivalence transformation, in contrast, generally only maps a possible solution of one underdetermined (unclosed) equation to a possible solution of another underdetermined (unclosed) equation (see e.g. Ovsiannikov (1982), Meleshko (1996), Ibragimov (2004), Bila (2011)). Hence, since any Lie-group invariance analysis on the infinite Friedmann-Keller hierarchy will be based on equivalence and not on symmetry groups, it is misleading and even ill-defined to construct invariant solutions of the unclosed Friedmann-Keller equations if no modelling assumptions on these set of equations are being priorly invoked (Frewer *et al.*, 2014a,b). Therefore, to state that "... a variety of classical and new scaling laws" were derived for which "it was shown that they are exact solutions of symmetry invariant type of the infinite dimensional series of MPC equations" (Avsarkisov *et al.*, 2014, p. 102) is more than misleading, because what to understand under "exact solution" when the construction process for generating solutions itself is arbitrary? Independent of whether a Lie group invariance analysis is employed or not, these "exact solutions" are nothing else but arbitrary and thus non-privileged solutions. They just arise from an *unclosed* infinite dimensional system which does not give any indication of whether these "exact results" are relevant solutions or not. For that, additional, *external* information is needed, e.g. as posing certain modelling assumptions to close the infinite system of equations.

The following study is organized as follows: Section 2 introduces the functional Hopf equation in physical space, from which the infinite Friedmann-Keller system of multi-point moments for spatially unconfined turbulence is then reduced. Section 3 investigates a lower-dimensional analogue to the system established in Section 2, but, which first will only feature the properties i) and ii) listed above. Instead of property iii), a backward recursively infinite system is first considered, in order to demonstrate that for such a case the non-uniqueness problem as discussed above cannot arise. Section 4 forms the key part of this study, in that an extended lower-dimensional analogue is considered which now will feature *all* properties i)-iii) of the

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<sup>†</sup>In Frewer *et al.* (2014a,b) it has been shown that this newly enlarged set of symmetries is even unphysical in that it leads to various inconsistencies with the underlying deterministic Navier-Stokes theory. Hence, besides the well-known classical symmetries of the Euler and Navier-Stokes equations (Fushchich *et al.*, 1993; Olver, 1993; Frisch, 1995; Andreev *et al.*, 1998), no new enlarged physical set of statistical symmetries exist.

system established in Section 2. This analogue will allow us to elucidate step by step all facets of the non-uniqueness problem which inherently characterizes the infinite Friedmann-Keller system of multi-point moments. Section 5 concludes and completes the investigation.

## 2. The Hopf equation and its induced MPC equations

The functional Hopf equation (Hopf, 1952; McComb, 1990) for the characteristic or moment generating functional  $\Phi = \Phi[\mathbf{y}(\mathbf{x}), t]$  in physical space is given by the *linear* differential equation

$$\frac{\partial \Phi}{\partial t} = \int y_\alpha^\perp(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x}, \quad (2.1)$$

where  $\mathbf{y}(\mathbf{x})$  is a real, integrable and time-independent auxiliary vector field being complementary to the ensemble of all admissible incompressible velocity fields  $\mathbf{u}(\mathbf{x})$  governed by the Navier-Stokes equations sampled at each time step  $t \geq 0$ . Since we consider an unbounded flow, we assume that the ensemble of fields  $\mathbf{y}(\mathbf{x})$  is vanishing sufficiently fast at infinity. The fields  $\mathbf{y}^\perp$  in equation (2.1) represent the transverse (solenoidal) part of  $\mathbf{y}(\mathbf{x})$  in order to eliminate the pressure terms which otherwise would arise in (2.1). We will use the transverse form as used in Hopf (1952) (see also Appendix A)

$$y_\alpha^\perp(\mathbf{x}) = y_\alpha(\mathbf{x}) - \partial_\alpha \varphi(\mathbf{x}), \quad (2.2)$$

with the scalar field

$$\varphi(\mathbf{x}) = - \int \frac{\nabla' \cdot \mathbf{y}(\mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}'. \quad (2.3)$$

Note that in order to warrant physical consistency, a mathematical solution of the Hopf equation (2.1) is only admitted if for all times the following conditions for the characteristic functional are fulfilled

$$\Phi^*[\mathbf{y}(\mathbf{x}), t] = \Phi[-\mathbf{y}(\mathbf{x}), t], \quad \Phi[0, t] = 1, \quad |\Phi[\mathbf{y}(\mathbf{x}), t]| \leq 1. \quad (2.4)$$

Now, when reducing the MPC equations from (2.1), one can either work in the representation of the full vector field  $\mathbf{y}$ , or in the decomposed representation of its transverse part  $\mathbf{y}^\perp$ . The former representation of the Hopf equation (2.1) takes the form (see Appendix B.1)

$$\begin{aligned} \frac{\partial \Phi}{\partial t} = & \int y_\alpha(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ & + i \int y_\alpha(\mathbf{x}) \left( \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial^2}{\partial x'_\beta \partial x'_\gamma} \frac{\delta^2 \Phi}{\delta y_\beta(\mathbf{x}') \delta y_\gamma(\mathbf{x}')} d^3 \mathbf{x}' d^3 \mathbf{x}, \end{aligned} \quad (2.5)$$

while the latter one takes the more concise form (see Appendix B.2)

$$\frac{\partial \Phi}{\partial t} = \int y_\alpha^\perp(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha^\perp(\mathbf{x})} \right) d^3 \mathbf{x}. \quad (2.6)$$

Note that both equations, (2.5) and (2.6), refer to the same characteristic functional  $\Phi$ , since under the defining relation (2.2) it transforms invariantly  $\Phi[\mathbf{y}(\mathbf{x}), t] = \Phi[\mathbf{y}^\perp(\mathbf{x}), t]$  (Hopf, 1952, p. 93). Keep in mind that equation (2.6) is a projection of equation (2.5) onto solenoidal vector fields, where if  $\nabla \cdot \mathbf{y} \neq 0$  then  $\nabla \cdot \mathbf{y}^\perp = 0$ . In other words, if the ensemble of all fields  $\mathbf{y}$  would be divergence-free, i.e.  $\nabla \cdot \mathbf{y} = 0$  (which in general is *not* assumed), then (2.5) is identical to (2.6), as can be readily observed in (2.5) after executing a partial integration with respect to  $\mathbf{x}$ .

Further note that both representations (2.5) and (2.6) of the Hopf equation (2.1) lead to the same set of MPC equations. It's obvious that the latter representation (2.6) is suited best

for any formal investigations as we are interested herein. It induces considerably simplified formal expressions for the MPC equations since no explicit pressure terms would surface as in representation (2.5). The corresponding pressure-dependent MPC equations, when needed, can then either be derived by using the defining relation (2.2) in the obtained results from the transverse-projected equation (2.6), or by directly deriving these from the full-field equation (2.5).

According to (Hopf, 1952, p. 100-101) the infinite Friedmann-Keller hierarchy of MPC equations can be generated from (2.6) by performing a Taylor expansion of  $\Phi[\mathbf{y}^\perp, t]$  around the functional point  $\mathbf{y}^\perp = \mathbf{0}$ :

$$\begin{aligned} \Phi[\mathbf{y}^\perp, t] = 1 + \int d^3\mathbf{x}_1 y_{\alpha_1}^\perp(\mathbf{x}_1) \frac{\delta\Phi[\mathbf{y}^\perp, t]}{\delta y_{\alpha_1}^\perp(\mathbf{x}_1)} \Big|_{\mathbf{y}^\perp=\mathbf{0}} \\ + \frac{1}{2!} \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 y_{\alpha_1}^\perp(\mathbf{x}_1) y_{\alpha_2}^\perp(\mathbf{x}_2) \frac{\delta^2\Phi[\mathbf{y}^\perp, t]}{\delta y_{\alpha_1}^\perp(\mathbf{x}_1) \delta y_{\alpha_2}^\perp(\mathbf{x}_2)} \Big|_{\mathbf{y}^\perp=\mathbf{0}} + \dots, \end{aligned} \quad (2.7)$$

which formally can be written as

$$\Phi[\mathbf{y}^\perp, t] = 1 + \Phi^1[\mathbf{y}^\perp, t] + \Phi^2[\mathbf{y}^\perp, t] + \dots, \quad (2.8)$$

where  $\Phi^n[\mathbf{y}^\perp, t]$  is a homogeneous polynomial functional of degree  $n$  in  $\mathbf{y}^\perp = \mathbf{y}^\perp(\mathbf{x})$ ,

$$\Phi^n[\mathbf{y}^\perp, t] = \int d^3\mathbf{x}_1 \dots d^3\mathbf{x}_n y_{\alpha_1}^\perp(\mathbf{x}_1) \dots y_{\alpha_n}^\perp(\mathbf{x}_n) K_{\alpha_1 \dots \alpha_n}^\perp(\mathbf{x}_1, \dots, \mathbf{x}_n, t), \quad (2.9)$$

and where the components of the kernel function  $\mathbf{K}_n^\perp = (K_{\alpha_1 \dots \alpha_n}^\perp)$  are given by

$$\begin{aligned} K_{\alpha_1 \dots \alpha_n}^\perp(\mathbf{x}_1, \dots, \mathbf{x}_n, t) &= \frac{1}{n!} \frac{\delta^n \Phi[\mathbf{y}^\perp, t]}{\delta y_{\alpha_1}^\perp(\mathbf{x}_1) \dots \delta y_{\alpha_n}^\perp(\mathbf{x}_n)} \Big|_{\mathbf{y}^\perp=\mathbf{0}} \\ &= \frac{1}{n!} \frac{\delta^n \Phi[\mathbf{y}, t]}{\delta y_{\alpha_1}(\mathbf{x}_1) \dots \delta y_{\alpha_n}(\mathbf{x}_n)} \Big|_{\mathbf{y}=\mathbf{0}} = K_{\alpha_1 \dots \alpha_n}(\mathbf{x}_1, \dots, \mathbf{x}_n, t). \end{aligned} \quad (2.10)$$

The last equality stems from the fact the functional derivatives stay invariant under the transformation (2.2) (see Appendix B.2); and since  $\mathbf{y}^\perp = \mathbf{0} \Leftrightarrow \mathbf{y} = \mathbf{0}$ , the kernel function constitutes itself as an invariant function, i.e.  $\mathbf{K}_n^\perp = \mathbf{K}_n$ . In Hopf (1952) this function is determined as the equal-time MPC function of the incompressible Navier-Stokes velocity field  $\mathbf{u}(\mathbf{x}, t)$

$$K_{\alpha_1 \dots \alpha_n}(\mathbf{x}_1, \dots, \mathbf{x}_n, t) = \frac{i^n}{n!} \langle u_{\alpha_1}(\mathbf{x}_1, t) \dots u_{\alpha_n}(\mathbf{x}_n, t) \rangle, \quad (2.11)$$

where  $\langle \cdot \rangle$  denotes the statistical ensemble average of the instantaneous and thus fluctuating velocity field evaluated at  $n$  different points. Upon inserting the power series of  $\Phi$  (2.8) into equation (2.6) and upon equating terms of equal degree on both sides, one finally obtains the infinite sequence of differential equations for the MPC functions

$$\frac{\partial \Phi^n}{\partial t} = \int y_\alpha^\perp(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi^{n+1}}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} + \nu \Delta \frac{\delta \Phi^n}{\delta y_\alpha^\perp(\mathbf{x})} \right) d^3\mathbf{x}, \quad \forall n \geq 1. \quad (2.12)$$

By taking the notation from Oberlack & Rosteck (2010), the above system can be evaluated to (see Appendix C.1)

$$\frac{\partial \mathbf{H}_n^\perp}{\partial t} + \sum_{i=1}^n \nabla_{\mathbf{x}_i} \cdot \hat{\mathbf{H}}_{i,n+1}^\perp - \nu \sum_{i=1}^n \Delta_{\mathbf{x}_i} \mathbf{H}_n^\perp = \mathbf{0}, \quad \forall n \geq 1, \quad (2.13)$$

where

$$\mathbf{H}_n^\perp(\mathbf{x}_1, \dots, \mathbf{x}_n, t) = \mathbf{H}_n(\mathbf{x}_1, \dots, \mathbf{x}_n, t) = \langle \mathbf{u}(\mathbf{x}_1, t) \otimes \dots \otimes \mathbf{u}(\mathbf{x}_n, t) \rangle, \quad (2.14)$$

is the instantaneous (equal-time) multi-point velocity correlation function of  $n$ -th order, and where

$$\hat{\mathbf{H}}_{i,n+1}^\perp(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n, \mathbf{x}_i, t) = \left( \lim_{\mathbf{x}_{n+1} \rightarrow \mathbf{x}_i} \mathbf{H}_{n+1}(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}, t) \right)^\perp, \quad 1 \leq i \leq n, \quad (2.15)$$

is not a  $(n+1)$ -point function, but only a lower dimensional  $n$ -point function of  $(n+1)$ -th order with the full-ranked transverse (solenoidal) property (see Appendix C.2)

$$\left( \nabla_{\mathbf{x}_1} \otimes \dots \otimes \nabla_{\mathbf{x}_i} \otimes \dots \otimes \nabla_{\mathbf{x}_n} \otimes \nabla_{\mathbf{x}_i} \right) \cdot \hat{\mathbf{H}}_{i,n+1}^\perp(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n, \mathbf{x}_i, t) = 0, \quad 1 \leq i \leq n. \quad (2.16)$$

The MPC equations (2.13) go along with the following natural incompressibility constraints

$$\left. \begin{aligned} \nabla_{\mathbf{x}_k} \cdot \mathbf{H}_n(\mathbf{x}_1, \dots, \mathbf{x}_n, t) &= \mathbf{0}, \quad \forall k \text{ between } 1 \leq k \leq n, \\ \nabla_{\mathbf{x}_k} \cdot \hat{\mathbf{H}}_{i,n+1}^\perp(\mathbf{x}_1, \dots, \mathbf{x}_l, \dots, \mathbf{x}_n, \mathbf{x}_l, t) &= \mathbf{0}, \quad \forall k, l \text{ between } 1 \leq (k, l) \leq n, \text{ for } k \neq l, \end{aligned} \right\} \quad (2.17)$$

due to the incompressibility constraint  $\nabla_{\mathbf{x}_k} \cdot \mathbf{u}(\mathbf{x}_k, t) = 0$  of the instantaneous velocity field  $\mathbf{u}(\mathbf{x}_k, t)$ , when evaluated at each point  $\mathbf{x} = \mathbf{x}_k$  (for all  $k = 1, \dots, n$ ) within the single physical domain  $\mathbf{x}$ . Note that, when decomposing the transverse fields (2.15) into their full fields (see Appendix C.2), the system (2.13)-(2.17) matches the infinite Friedmann-Keller hierarchy of MPC equations as derived in Oberlack & Rosteck (2010) for the instantaneous case.

Now, if we collect the key properties of the combined system (2.1) and (2.13), the most important thing about it is that it is linear. Both the higher level Hopf equation (2.1) as well as its induced lower level system of MPC equations (2.13) constitute a linear system of equations. The former as a singly closed functional equation, while the latter as an infinite hierarchy of PDEs. As we will see during this study, a crucial aspect of the considered hierarchy (2.13) is that it is defined forward recursively. To illustrate the basic problems from which such an infinite system suffers, we will investigate the following lower-dimensional analogous system for a spatially one-dimensional field  $u = u(x, t)$  with its induced moments  $u_n = u_n(t)$

$$\partial_t u = \partial_x^2 u - \lambda \cdot x^2 u, \quad (2.18)$$

$$\frac{du_n}{dt} = n \cdot (n-1) \cdot u_{n-2} - \lambda \cdot u_{n+2}, \quad \text{where } u_n(t) = \int_{-\infty}^{\infty} x^n \cdot u(x, t) dx, \quad n \geq 0, \quad (2.19)$$

which features all key properties of the original combined system: The linear and non-autonomous property of the higher level functional Hopf equation (2.1) is featured by the higher level PDE (2.18), from which the lower level ordinary differential (ODE) moment equations (2.19) are induced, which themselves again feature the linear, autonomous and forward recursive property of the infinite Friedmann-Keller hierarchy of MPC equations (2.13) which again emerge from the higher level Hopf equation<sup>†</sup> (see Appendix D for the derivation of (2.19) from (2.18)).

When considering in particular the original system of moments (2.13), the foremost aim, of course, is to find certain solutions, at least some asymptotic solutions, e.g. as attempted by Oberlack et al. in applying a Lie group based symmetry group analysis directly on system (2.13) in order to generate certain specific invariant solutions which then should function as first

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<sup>†</sup>Note that the incompressibility constraints (2.17) cannot be mapped down to a one-dimensional system. Hence, the incompressibility property of the original MPC system of equations (2.13) is not featured by its analogous infinite system (2.19). But this is no drawback for the issues to be addressed in this study, since the incompressibility constraints are irrelevant and thus do not influence the conclusions made herein.



principle scaling laws within some asymptotic flow regime (Oberlack & Rosteck, 2010; Oberlack & Zieleniewicz, 2013; Avsarkisov *et al.*, 2014; Waclawczyk *et al.*, 2014; Oberlack *et al.*, 2014).

Our claim is that system (2.13), along with its incompressibility constraints (2.17), is unclosed, even if all infinite equations are formally considered. Hence such an approach as performed by Oberlack *et al.* is misleading when a symmetry analysis is only carried out for the statistical system of moments without additionally incorporating the underlying deterministic set of equations. In other words, to generate any kind of solutions directly from (2.13) is not well defined if no prior modelling assumptions in accord with the deterministic Navier-Stokes equations are made to close the statistical system of moments.

The reason why system (2.13) is unclosed, even if all infinite equations are considered, is basically twofold: The first issue lies within the non-symmetric and thus incomplete coupling between the equations. For each order, the unknown moment  $\hat{\mathbf{H}}_{i,n+1}^\perp$  is not directly coupled to the next higher order equation. Although all components of that moment can be uniquely constructed from the higher dimensional moment  $\mathbf{H}_{n+1}$  once they are known, which is formally denoted as (2.15), the necessary inverse construction, however, fails. Hence, since (2.15) is a non-invertible construction, i.e. since  $\mathbf{H}_{n+1}$  cannot be uniquely constructed from  $\hat{\mathbf{H}}_{i,n+1}^\perp$ , these latter moments are to be identified as unclosed functions in system (2.13) as they do not directly enter the next higher order correlation equation. For each order  $n$  in the hierarchy the total number of dynamical equations (2.13), along with the continuity constraint equations (2.17), is therefore always less than the total number of unknown functions. In total this just reflects the classical closure problem of turbulence for the moments which cannot be bypassed by simply establishing the formal connection (2.15) — for a more detailed discussion on this issue, please refer to Appendix A & C in Frewer *et al.* (2014a) and Section 3 in Frewer *et al.* (2014b).

The second and more fundamental issue of why the infinite hierarchy (2.13) is unclosed, and which forms the focus of this study, is that the hierarchy is defined *forward* recursively. As I will elucidate at the example of the lower-dimensional analogue (2.19) to system (2.13), an infinite forward-recursively defined hierarchy of equations always leads to a non-uniqueness problem in its solution manifold — even when sufficient initial conditions are posed, a system such as (2.13), when stated as a well-posed initial value problem, still does not return a unique solution. However, this problem does *not* arise if any infinite hierarchy is considered which is defined *backward* recursively. Hence, instead of the full system (2.18)-(2.19), I will first investigate the solution properties of the subsystem  $\lambda = 0$ , which will be done in the next section, thus leading us to an infinite hierarchy of moment equations which first will be defined backward recursively.

### 3. Example for an infinite backward differential recurrence relation

#### — A closed system with a unique general solution manifold

In this section we will analyze the combined system (2.18)-(2.19) for  $\lambda = 0$ , which hence results to a study of the usual one-dimensional diffusion equations and its moments. Consider first the higher level PDE (2.18) by stating it as the following well-posed Cauchy problem

$$\partial_t u = \partial_x^2 u, \quad \text{with } u(x, 0) = \phi(x), \quad (3.1)$$

where we assume that the function  $u = u(x, t)$  is decaying sufficiently fast at infinity, i.e.

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \partial_x u(x, t) = 0, \quad \forall t \geq 0, \quad (3.2)$$

and where, for convenience, we want to assume that the initial function  $\phi$  is normalized to

$$\int_{-\infty}^{\infty} \phi(x) dx = 1. \quad (3.3)$$

Then the initial value problem (3.1) has the *unique* solution (see e.g. Polyanin (2002))

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4t}} \phi(x') dx', \quad \text{for } t \geq 0. \quad (3.4)$$

If we would choose as a initial condition, for example, the Gaussian distribution

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \sigma, \mu \in \mathbb{R}, \quad \sigma > 0, \quad (3.5)$$

then the solution (3.4) would read

$$u(x, t) = \frac{1}{\sqrt{2\pi(2t + \sigma^2)}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{2t + \sigma^2}}, \quad t \geq 0, \quad (3.6)$$

with its associated moments uniquely given as (shown here only up to third order)

$$n = 0: \quad u_0(t) = \int_{-\infty}^{\infty} x^0 \cdot u(x, t) dx = 1, \quad (3.7)$$

$$n = 1: \quad u_1(t) = \int_{-\infty}^{\infty} x^1 \cdot u(x, t) dx = \mu, \quad (3.8)$$

$$n = 2: \quad u_2(t) = \int_{-\infty}^{\infty} x^2 \cdot u(x, t) dx = 2t + \mu^2 + \sigma^2, \quad (3.9)$$

$$n = 3: \quad u_3(t) = \int_{-\infty}^{\infty} x^3 \cdot u(x, t) dx = \mu(6t + \mu^2 + 3\sigma^2). \quad (3.10)$$

Now, let's consider the corresponding moment equations when they are independently derived as an infinitely coupled hierarchy from the higher-level Cauchy problem (3.1) (see Appendix D)

$$\frac{du_n}{dt} = n \cdot (n-1) \cdot u_{n-2}, \quad \text{with } u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) dx, \quad n \geq 0. \quad (3.11)$$

Our aim is to investigate in how far the above infinite system of first order ODEs represents a well-posed initial value-problem for the moments  $u_n = u_n(t)$ . The first positive observation is that to each unknown function  $u_n$  one can bijectively associate an initial condition  $u_n(0)$  to it, since any first order ODE only requires a single initial condition in order to provide a unique solution. The next step is to derive the general solution, for which it is helpful to recognize that since the infinite hierarchy in (3.11) is defined *backward recursively of an order two*, it can be rewritten into the following equivalent form

$$\left. \begin{aligned} \frac{du_0(t)}{dt} &= 0; & \frac{d^{k+1}u_{2k+2}(t)}{dt^{k+1}} &= (2k+2)! \cdot u_0(t), \quad k \geq 0, \\ \frac{du_1(t)}{dt} &= 0; & \frac{d^l u_{2l+1}(t)}{dt^l} &= (2l+1)! \cdot u_1(t), \quad l \geq 1. \end{aligned} \right\} \quad (3.12)$$



This system can then be uniquely integrated to give the *general* solution

$$\left. \begin{aligned} u_0(t) &= c_0, \\ u_{2k+2}(t) &= c_{2k+2} + \sum_{j=1}^k \frac{q_{k,j}^{(1)}}{j!} t^j \\ &\quad + (2k+2)! \int_0^{t_k=t} \int_0^{t_{k-1}} \cdots \int_0^{t_0} u_0(t') dt' dt_0 \cdots dt_{k-1}, \quad k \geq 0, \\ u_1(t) &= c_1, \\ u_{2l+1}(t) &= c_{2l+1} + \sum_{j=1}^{l-1} \frac{q_{l,j}^{(2)}}{j!} t^j \\ &\quad + (2l+1)! \int_0^{t_l=t} \int_0^{t_{l-1}} \cdots \int_0^{t_1} u_1(t') dt' dt_1 \cdots dt_{l-1}, \quad l \geq 1, \end{aligned} \right\} \quad (3.13)$$

with the expansion coefficients given as

$$\left. \begin{aligned} q_{k,j}^{(1)} &= \frac{(2k+2)!}{(2k+2-2j)!} c_{2k+2-2j}, \quad k \geq 0; \quad 0 \leq j \leq k, \\ q_{l,j}^{(2)} &= \frac{(2l+1)!}{(2l+1-2j)!} c_{2l+1-2j}, \quad l \geq 1; \quad 0 \leq j \leq l-1, \end{aligned} \right\} \quad (3.14)$$

where all  $c_n$  for  $n \geq 0$  are arbitrary integration constants. Hence, the second positive observation is that the *unrestricted* system of equations in (3.11)

$$\frac{du_n(t)}{dt} = n \cdot (n-1) \cdot u_{n-2}(t), \quad n \geq 0, \quad (3.15)$$

provides a general solution (3.13) which only involves arbitrary constants, i.e. the unrestricted system (3.15) provides a *unique* general solution and therefore it ultimately represents a formally closed system of equations. Because, when restricting this system to the underlying PDE's initial condition  $u(x,0) = \phi(x)$  as given in (3.1), which for the infinite ODE system takes the integral form  $u_n(0)$  as given in (3.11), will turn the general solution (3.13) into a unique and fully determined solution, where the integration constants are then given by

$$c_n = u_n(0), \quad n \geq 0. \quad (3.16)$$

By choosing for example again the specific initial condition (3.5), the general solution (3.13) will give exactly the expressions (3.7)-(3.10) for the moments, which, as we know, were directly derived from the higher-level PDE solution (3.4) of the well-posed Cauchy problem (3.1). Hence, since the *unrestricted* infinite system of ODEs (3.15) constitutes a formally closed system of equations, the *restricted* system (3.11) expresses a well-posed initial value problem.

That (3.15) constitutes a formally closed system is also supported when performing a Lie-group based symmetry analysis (see e.g. Stephani (1989); Olver (1993); Bluman & Kumei (1996); Ibragimov (1994); Hydon (2000)). The infinite and coupled system (3.15) admits the

following complete and unique set of Lie-point symmetries<sup>†</sup>

$$\left. \begin{aligned}
 S_1 : X_1 &= \partial_t, \\
 S_2 : X_2 &= t\partial_t - 2u_0\partial_{u_0} - u_2\partial_{u_2} + \cdots + (n-2) \cdot u_{2n}\partial_{u_{2n}} + \cdots \\
 &\quad - 2u_1\partial_{u_1} - u_3\partial_{u_3} + \cdots + (n-2) \cdot u_{2n+1}\partial_{u_{2n+1}} + \cdots, \\
 S_3 : X_3 &= u_0\partial_{u_0} + u_2\partial_{u_2} + \cdots + u_{2n}\partial_{u_{2n}} + \cdots, \\
 S_4 : X_4 &= u_1\partial_{u_1} + u_3\partial_{u_3} + \cdots + u_{2n+1}\partial_{u_{2n+1}} + \cdots, \\
 S_5 : X_5 &= u_1\partial_{u_0} + \frac{1}{3}u_3\partial_{u_2} + \cdots + \frac{1}{2n+1}u_{2n+1}\partial_{u_{2n}} + \cdots, \\
 S_6 : X_6 &= u_0\partial_{u_1} + 3u_2\partial_{u_3} + \cdots + (2n+1) \cdot u_{2n}\partial_{u_{2n+1}} + \cdots, \\
 S_{2n}^\infty : X_{2n}^\infty &= \partial_{u_{2n}} + \frac{(2n+2)!}{(2n)!}t\partial_{u_{2n+2}} + \frac{(2n+4)!}{2!(2n)!}t^2\partial_{u_{2n+4}} + \cdots \\
 &\quad + \cdots + \frac{(2n+2m)!}{m!(2n)!}t^m\partial_{u_{2n+2m}} + \cdots, \quad m \geq 0, \\
 S_{2n+1}^\infty : X_{2n+1}^\infty &= \partial_{u_{2n+1}} + \frac{(2n+3)!}{(2n+1)!}t\partial_{u_{2n+3}} + \frac{(2n+5)!}{2!(2n+1)!}t^2\partial_{u_{2n+5}} + \cdots \\
 &\quad + \cdots + \frac{(2n+2m+1)!}{m!(2n+1)!}t^m\partial_{u_{2n+2m+1}} + \cdots, \quad m \geq 0,
 \end{aligned} \right\} \quad (3.17)$$

where  $X_k$  in each case is the infinitesimal generator of the symmetry  $S_k$ . These generators form a *closed* Lie algebra as shown in the commutator table below. Since a Lie algebra inherently defines a linear vector space, any linear combination of generators (3.17) forms again a symmetry generator of the considered system (3.15). And, due to the large variety of symmetries admitted, it is possible to combine these symmetry generators (3.17) into several independent symmetries such that they are compatible with the posed initial condition given in (3.11). Altogether three such compatible symmetries can be constructed from (3.17):

$$\left. \begin{aligned}
 S_1^\phi : X_1^\phi &= X_2 - \sum_{k=0}^{\infty} (k-2) \left( u_{2k}(0) \cdot X_{2k}^\infty + u_{2k+1}(0) \cdot X_{2k+1}^\infty \right), \\
 S_2^\phi : X_2^\phi &= X_3 + X_4 - \sum_{k=0}^{\infty} \left( u_{2k}(0) \cdot X_{2k}^\infty + u_{2k+1}(0) \cdot X_{2k+1}^\infty \right), \\
 S_3^\phi : X_3^\phi &= X_5 + X_6 - \sum_{k=0}^{\infty} \left( \frac{1}{2k+1} \cdot u_{2k+1}(0) \cdot X_{2k}^\infty + (2k+1) \cdot u_{2k}(0) \cdot X_{2k+1}^\infty \right).
 \end{aligned} \right\} \quad (3.18)$$

When generating invariant solutions from (3.18)<sup>‡</sup>, all three symmetries independently will

<sup>†</sup>The explicit result for the symmetries (3.17) was obtained by augmenting the considered system (3.15) to an extended system which also includes all admissible differential consequences of (3.15). Otherwise one runs the high risk of performing an inconsistent symmetry analysis. Note that the symmetry corresponding to the linear superposition principle has not been included in (3.17), since it cannot be taken to directly construct group invariant solutions as we intend to do here.

<sup>‡</sup>Note that the determination of an invariant function from any of the three symmetries (3.18) is not sufficient to be a solution of the underlying system (3.11). In general such an invariant function is endowed with integration constants which can only be uniquely determined by plugging the function back into its determining system of equations, i.e. here, back into system (3.11). But note that if one of these integration constants can not be equated consistently, then the invariant function does not constitute a solution.

$[\cdot, \cdot]$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_{2n}^\infty$	$X_{2n+1}^\infty$
$X_1$	0	$X_1$	0	0	0	0	$\frac{(2n+2)!}{(2n)!} X_{2n+2}^\infty$	$\frac{(2n+3)!}{(2n+1)!} X_{2n+3}^\infty$
$X_2$		0	0	0	0	0	$-(n-2)X_{2n}^\infty$	$-(n-2)X_{2n+1}^\infty$
$X_3$			0	0	$-X_5$	$X_6$	$-X_{2n}^\infty$	0
$X_4$				0	$X_5$	$-X_6$	0	$-X_{2n+1}^\infty$
$X_5$					0	$X_3 + X_4$	0	$-\frac{1}{2n+1} X_{2n}^\infty$
$X_6$						0	$-(2n+1)X_{2n+1}^\infty$	0
$X_{2n}^\infty$							0	0
$X_{2n+1}^\infty$								0

Table 1: Commutator table for the generators (3.17), where  $[X_i, X_j] = -[X_j, X_i] := X_i X_j - X_j X_i$ .

finally give the same *unique* solution set (3.13), with  $c_n = u_n(0)$  (3.16), as when solving directly the system of equations (3.11).

#### 4. Example for an infinite forward differential recurrence relation

##### — An unclosed system with a non-unique general solution manifold

In this section we will investigate an infinite system which is not closed; which yet not even formally can be regarded as closed. By analyzing the combined system (2.18)-(2.19) for  $\lambda = 1$ , we are now obtaining a one-dimensional system which, in contrast to the previously in Section 3 considered system, now features *all* key properties of the higher level induced (Hopf-equation induced) infinite Friedmann-Keller hierarchy of MPC equations (2.13), in particular where the crucial property of being a forward recursively defined relation is now included. And it is solely this property which now turns the problem into an unclosed problem.

As it was done in the previous section for the specification  $\lambda = 0$ , let us also for  $\lambda = 1$  first consider the higher level PDE (2.18) as the following well-posed Cauchy problem

$$\partial_t u = \partial_x^2 u - x^2 u, \text{ for } t \geq 0, \text{ with } u(x, 0) = \phi(x), \quad (4.1)$$

where we again assume that the function  $u = u(x, t)$  is decaying sufficiently fast at infinity, i.e.

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \partial_x u(x, t) = 0, \quad \forall t \geq 0, \quad (4.2)$$

and where also again, for convenience, we want to assume that the initial function  $\phi$  is normalized to one

$$\int_{-\infty}^{\infty} \phi(x) dx = 1. \quad (4.3)$$

To solve this initial value problem (4.1) it is necessary to realize that the following nonlinear

point transformation (Polyanin, 2002)<sup>†</sup>

$$\tilde{t} = \frac{1}{4} \cdot (e^{4t} - 1), \quad \tilde{x} = x \cdot e^{2t}, \quad \tilde{u} = u \cdot e^{-\frac{1}{2}x^2 - t}, \quad (4.4)$$

which has the unique inverse transformation

$$t = \frac{1}{4} \ln(1 + 4\tilde{t}), \quad x = \frac{\tilde{x}}{\sqrt{1 + 4\tilde{t}}}, \quad u = \tilde{u} \cdot \sqrt[4]{1 + 4\tilde{t}} \cdot e^{\frac{1}{2} \cdot \frac{\tilde{x}^2}{1 + 4\tilde{t}}}, \quad (4.5)$$

maps the original Cauchy problem (4.1) into the following Cauchy problem for the standard diffusion equation with constant coefficients:

$$\partial_{\tilde{t}} \tilde{u} = \partial_{\tilde{x}}^2 \tilde{u}, \quad \text{for } \tilde{t} \geq 0, \quad \text{with } \tilde{u}(\tilde{x}, 0) = \phi(\tilde{x}) \cdot e^{-\frac{1}{2}\tilde{x}^2}. \quad (4.6)$$

Important to note here is that the initial time  $t = 0$  as well as the relevant time range  $t \in [0, \infty)$  both get invariantly mapped to  $\tilde{t} = 0$  and  $\tilde{t} \in [0, \infty)$  respectively. Hence, the unique solution of the transformed Cauchy problem (4.6) is thus again given by (3.4), but now in the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \frac{1}{\sqrt{4\pi\tilde{t}}} \int_{-\infty}^{\infty} e^{-\frac{(\tilde{x} - \tilde{x}')^2}{4\tilde{t}}} \phi(\tilde{x}') e^{-\frac{1}{2}\tilde{x}'^2} d\tilde{x}', \quad \text{for } \tilde{t} \geq 0, \quad (4.7)$$

which then, according to transformation (4.4), leads to the *unique* solution for the original Cauchy problem (4.1)

$$u(x, t) = \frac{e^{\frac{1}{2}x^2 + t}}{\sqrt{\pi(e^{4t} - 1)}} \int_{-\infty}^{\infty} e^{-\frac{(e^{2t}x - x')^2}{e^{4t} - 1}} \phi(x') e^{-\frac{1}{2}x'^2} dx', \quad \text{for } t \geq 0. \quad (4.8)$$

If we again would choose as a initial condition, for example, the Gaussian distribution (3.5)

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}, \quad \sigma, \mu \in \mathbb{R}, \quad \sigma > 0, \quad (4.9)$$

then the solution (4.8) would have the explicit form

$$u(x, t) = \frac{1}{\sqrt{\pi(\sigma^2 + e^{4t}(1 + \sigma^2) - 1)}} e^{t + \frac{x^2}{2} - \frac{e^{4t}(\mu^2 + 2x^2(1 + \sigma^2)) - 4e^{2t}\mu x + \mu^2}{2(\sigma^2 + e^{4t}(1 + \sigma^2) - 1)}}, \quad t \geq 0. \quad (4.10)$$

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<sup>†</sup>This continuous point transformation is *not* a group transformation, as it neither includes a group parameter nor does it include the unique continuously connected identity transformation from which any infinitesimal mapping can emanate.

Its moments for *all*  $t \geq 0$  are uniquely given as (shown here again only up to third order)

$$n = 0 : \quad u_0(t) = \int_{-\infty}^{\infty} x^0 \cdot u(x, t) dx = \frac{\sqrt{2} e^{t - \frac{(-1+e^{4t})\mu^2}{2(1-\sigma^2+e^{4t}(1+\sigma^2))}}}{\sqrt{1-\sigma^2+e^{4t}(1+\sigma^2)}}, \quad (4.11)$$

$$n = 1 : \quad u_1(t) = \int_{-\infty}^{\infty} x^1 \cdot u(x, t) dx = \frac{2\mu\sqrt{2} e^{3t - \frac{(-1+e^{4t})\mu^2}{2(1-\sigma^2+e^{4t}(1+\sigma^2))}}}{\sqrt{(1-\sigma^2+e^{4t}(1+\sigma^2))^3}}, \quad (4.12)$$

$$n = 2 : \quad u_2(t) = \int_{-\infty}^{\infty} x^2 \cdot u(x, t) dx = \frac{\sqrt{2} e^{t - \frac{(-1+e^{4t})\mu^2}{2(1-\sigma^2+e^{4t}(1+\sigma^2))}}}{\sqrt{(1-\sigma^2+e^{4t}(1+\sigma^2))^5}} \cdot \left(4e^{4t}\mu^2 - (-1+\sigma^2)^2 + e^{8t}(1+\sigma^2)^2\right), \quad (4.13)$$

$$n = 3 : \quad u_3(t) = \int_{-\infty}^{\infty} x^3 \cdot u(x, t) dx = \frac{2\mu\sqrt{2} e^{3t - \frac{(-1+e^{4t})\mu^2}{2(1-\sigma^2+e^{4t}(1+\sigma^2))}}}{\sqrt{(1-\sigma^2+e^{4t}(1+\sigma^2))^7}} \cdot \left(4e^{4t}\mu^2 - 3(-1+\sigma^2)^2 + 3e^{8t}(1+\sigma^2)^2\right). \quad (4.14)$$

Now, let's consider the corresponding moment equations when they are independently derived from the higher-level Cauchy problem (4.1) (see Appendix D)

$$\frac{du_n}{dt} = n \cdot (n-1) \cdot u_{n-2} - u_{n+2}, \quad \text{with } u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) dx, \quad n \geq 0. \quad (4.15)$$

Our aim is again to investigate in how far the above infinite system of first order ODEs represents a well-posed initial value-problem for the moments  $u_n = u_n(t)$ . As it was also already noticed for the system in the previous section, the first positive observation is that in this case too, one can bijectively associate an initial condition  $u_n(0)$  to each unknown function  $u_n$ . But the next step, deriving the general solution of (4.15), stands in clear contrast to the previous section: The system's general solution is *not* uniquely specified, i.e. even if sufficient initial conditions are imposed, system (4.15) still does not offer a unique solution. In clear contrast, of course, to its associated higher level PDE system (4.1), which, as a Cauchy problem, is well-posed by providing the unique solution (4.10) with its unique moments (4.11)-(4.14). To see this problem explicitly, it is helpful to first recognize that the infinite ODE system (4.15) can be rewritten into the following equivalent and already solved form<sup>†</sup>

$$\left. \begin{aligned} u_{2k+2}(t) &= (-1)^{k+1} \sum_{i=0}^{\infty} A_i^{(1)}(k) \frac{d^{k+1-2i}}{dt^{k+1-2i}} u_0(t), \quad k \geq 0, \\ u_{2l+1}(t) &= (-1)^l \sum_{j=1}^{\infty} A_j^{(2)}(l) \frac{d^{l+2-2j}}{dt^{l+2-2j}} u_1(t), \quad l \geq 1, \end{aligned} \right\} \quad (4.16)$$

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<sup>†</sup>In the following we agree on the definitions that  $\frac{d^0}{dt^0} = 1$ ,  $\frac{d^{q<0}}{dt^q} = 0$ , and  $\sum_{i=0}^{q<0} = 0$ .

where the coefficients  $A_i^{(1)}(k)$  and  $A_j^{(2)}(l)$  are recursively defined as:

$$\left. \begin{aligned} A_i^{(1)}(k) &= \sum_{q=0}^{k-(2i-1)} (2k-2q) \cdot (2k-1-2q) \cdot A_{i-1}^{(1)}(k-2-q), \quad i \geq 1, \quad k \geq 0, \\ A_j^{(2)}(l) &= \sum_{q=1}^{l-(2j-3)} (2l-2q) \cdot (2l+1-2q) \cdot A_{j-1}^{(2)}(l-1-q), \quad j \geq 2, \quad l \geq 1, \end{aligned} \right\} \quad (4.17)$$

with the initial seeds:  $A_0^{(1)}(-1) = 1$ ,  $A_0^{(1)}(k) = 1$ ,  $\forall k \geq 0$ , and  $A_1^{(2)}(0) = 1$ ,  $A_1^{(2)}(l) = 1$ ,  $\forall l \geq 1$ , respectively. In contrast to the integrated *general* solution (3.13) of the previously considered *unrestricted* system (3.15), we see that the degree of underdeterminedness in the above derived *general* solution (4.16) is fundamentally different and higher than in (3.13). Instead of integration constants  $c_n$ , we now have two integration functions  $u_0(t)$  and  $u_1(t)$  which can be chosen freely. Their (arbitrary) specification will then determine all other solutions for  $k \geq 0$  and  $l \geq 1$  according to (4.16). The reason for having two free functions and not infinitely many free constants is that the *unrestricted* system of equations in (4.15)

$$\frac{du_n(t)}{dt} = n \cdot (n-1) \cdot u_{n-2}(t) - u_{n+2}(t), \quad n \geq 0, \quad (4.18)$$

defines a *forward* recurrence relation (of order two)<sup>†</sup> that needs *not* to be integrated in order to determine its general solution, while system (3.15), in contrast, defines a *backward* recurrence relation (of order two) which needs to be integrated to yield its general solution.

To explicitly demonstrate that (4.16) is not a *unique* general solution, we have to impose the initial condition  $u(x, 0) = \phi(x)$  for the corresponding moments as given in (4.15). The easiest and most apparent way to perform this implementation would be in a first choice to choose the two arbitrary functions  $u_0(t)$  and  $u_1(t)$  as analytical functions, since they straightforwardly allow for an expansion into power series

$$u_0(t) = \sum_{p=0}^{\infty} \frac{c_p^{(1)}}{p!} t^p, \quad u_1(t) = \sum_{p=0}^{\infty} \frac{c_p^{(2)}}{p!} t^p, \quad (4.19)$$

where  $c_p^{(1)}$  and  $c_p^{(2)}$  are two different infinite sets of constant expansion coefficients. By inserting this Ansatz into the general solution (4.16) and imposing the initial conditions as given in (4.15) will then uniquely specify these coefficients in a recursive manner as

$$\left. \begin{aligned} c_p^{(1)} &= 0, \quad p < 0; \quad c_p^{(1)} = (-1)^p \cdot u_{2p}(0) - \sum_{r=1}^{\infty} A_r^{(1)}(p-1) \cdot c_{p-2r}^{(1)}, \quad p \geq 0, \\ c_p^{(2)} &= 0, \quad p < 0; \quad c_p^{(2)} = (-1)^p \cdot u_{2p+1}(0) - \sum_{r=1}^{\infty} A_{r+1}^{(2)}(p) \cdot c_{p-2r}^{(2)}, \quad p \geq 0. \end{aligned} \right\} \quad (4.20)$$

Indeed, the two functions (4.19) with the above determined coefficients (4.20) form the analytical part of the corresponding unique PDE moment solutions (4.11)-(4.14) when choosing the initial-condition function  $\phi$  as the Gaussian distribution (4.9). In other words, if the unique PDE moment solutions (4.11)-(4.14) were Taylor expanded around  $t = 0$ , they would exactly yield the first four power series solutions (4.19) of the associated infinite ODE system (4.15). But, the obvious problem is that the underlying PDE solutions (4.11)-(4.14) do *not* constitute

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<sup>†</sup>The order of the recurrence relation is defined relative to the differential operator.



analytical functions in the independent variable  $t$ , i.e. the Taylor expansions of the unique solution functions (4.11)-(4.14) do not converge on the whole temporal domain, but only in the following limited range (dependent, of course, on the value of the variance  $\sigma^2$ )

$$\left. \begin{aligned} 0 < \sigma^2 < 1: \quad 0 \leq t < \frac{1}{4} \sqrt{\pi^2 + \left[ \ln \left( \frac{1 - \sigma^2}{1 + \sigma^2} \right) \right]^2}, \\ \sigma^2 = 1: \quad 0 \leq t < \infty, \\ \sigma^2 > 1: \quad 0 \leq t < \frac{1}{4} \sqrt{(2\pi)^2 + \left[ \ln \left( \frac{\sigma^2 - 1}{\sigma^2 + 1} \right) \right]^2}. \end{aligned} \right\} \quad (4.21)$$

That means, our initial assumption that the first two ODE solutions  $u_0(t)$  and  $u_1(t)$  are analytical functions on the global and unlimited scale  $t \in [0, \infty)$ , for all values of  $\sigma^2$ , is thus not correct. Only for a very limited range, where  $\sigma^2 \sim 1$ , this functional choice (4.19) is valid. But, if we don't know the full scale PDE solutions (4.11)-(4.14) beforehand, how then to choose these two unknown functions  $u_0(t)$  and  $u_1(t)$  for the infinite ODE system (4.16)? The clear answer is that there is no way without invoking a prior modelling assumption on the ODE system itself. Even if we would choose specific functions  $f_0(t)$  and  $f_1(t)$ , which for  $u_0(t)$  and  $u_1(t)$  are valid on any larger scale than the limited analytical Ansatz (4.19), we still have the problem that this particular solution choice is not unique, because one can always add to this choice certain independent functions which give no contributions when evaluated at the initial point  $t = 0$ . For example, if  $u_0(t) = f_0(t)$  and  $u_1(t) = f_1(t)$ , and if both functions  $f_0$  and  $f_1$  satisfy the given initial conditions at  $t = 0$ , then

$$u_0(t) = f_0(t) + \psi_0(t) \cdot e^{-\frac{\gamma_0^2}{t^2}}, \quad u_1(t) = f_1(t) + \psi_1(t) \cdot e^{-\frac{\gamma_1^2}{t^2}}, \quad (4.22)$$

is also a possible solution choice which satisfies the same initial conditions, where  $\psi_0(t)$  and  $\psi_1(t)$  are again (new) arbitrary functions, with the only restriction that, at the initial point  $t = 0$ , they have to increase slower than  $e^{\gamma_0^2/t^2}$  and  $e^{\gamma_1^2/t^2}$  respectively.

Note that  $u_0(t)$  and  $u_1(t)$  are not privileged in the sense that only these functions can be chosen arbitrarily. Any two functions  $u_{n^*}(t)$  and  $u_{m^*}(t)$  in the hierarchy (4.18) can be chosen freely, where  $n = n^*$  and  $n = m^*$  are some arbitrary but fixed orders in this hierarchy such that  $n^* \neq m^* + 2k$ ,  $\forall k \in \mathbb{Z}$ . For example, for the choice  $n^* = 2$  and  $m^* = 3$  in (4.18) its underdetermined general solution will take the form

$$\left. \begin{aligned} u_0(t) &= u_0(0) - \int_0^t u_2(t') dt', \\ u_{2k+2}(t) &= (-1)^{k+1} \sum_{i=0}^{\infty} A_i^{(1)}(k) \left[ \delta_{0,k+1-2i} \left( u_0(0) - \int_0^t u_2(t') dt' \right) - \frac{d^{k-2i}}{dt^{k-2i}} u_2(t) \right], \quad k \geq 1, \\ u_1(t) &= u_1(0) - \int_0^t u_3(t') dt', \\ u_{2l+1}(t) &= (-1)^l \sum_{j=1}^{\infty} A_j^{(2)}(l) \left[ \delta_{0,l+2-2j} \left( u_1(0) - \int_0^t u_3(t') dt' \right) - \frac{d^{l+1-2j}}{dt^{l+1-2j}} u_3(t) \right], \quad l \geq 2, \end{aligned} \right\} \quad (4.23)$$

where now, compared to the alternative general solution (4.16), the functions  $u_2(t)$  and  $u_3(t)$  are the unclosed terms, instead of  $u_0(t)$  and  $u_1(t)$ .

That the infinite system (4.18) is underdetermined and thus unclosed is also supported when performing an invariance analysis on it. To transform system (4.18) invariantly, complete

arbitrariness exists in that three arbitrary functions are available in order to perform the transformation: One for the independent variable  $t$  and two for any arbitrary but fixed chosen dependent variables  $u_{n^*}$  and  $u_{m^*}$ , such that  $n^* \neq m^* + 2k$ ,  $\forall k \in \mathbb{Z}$ , i.e. where in effect the transformation of two functions  $u_{n^*} = u_{n^*}(t)$  and  $u_{m^*} = u_{m^*}(t)$  can be chosen absolutely freely. Note that the outcome of such an (unclosed) invariance analysis does not result into symmetry transformations, but only into weaker *equivalence* transformations which invariantly only map between unclosed systems (see also the discussion partly done in the Introduction; for more details on this issue, see Frewer *et al.* (2014a,b) and the references therein).

Hence, when choosing, for example,  $n^* = 0$  and  $m^* = 1$ , then infinitely many functionally independent equivalence transformations of the following form are admitted by system (4.18)

$$\begin{aligned} E_{(\xi, \eta_{u_0}, \eta_{u_1})}^\infty : X = & \xi \partial_t + \eta_{u_0} \partial_{u_0} + \eta_{u_1} \partial_{u_1} \\ & + \eta_{u_2}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3) \partial_{u_2} \\ & + \eta_{u_3}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3) \partial_{u_3} \\ & + \cdots + \eta_{u_{2n}}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3, \dots, u_{2n}, u_{2n+1}) \partial_{u_{2n}} \\ & + \eta_{u_{2n+1}}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3, \dots, u_{2n}, u_{2n+1}) \partial_{u_{2n+1}} + \cdots, \end{aligned} \quad (4.24)$$

where  $\xi = \xi(t, u_0, u_1)$ ,  $\eta_{u_0}(t, u_0, u_1)$  and  $\eta_{u_1}(t, u_0, u_1)$  are arbitrary, free choosable functions. Consequently, the infinite set of invariant transformations (4.24) do *not* form a closed Lie algebra (in contrast to the infinite set (3.17) of the previously considered system). In particular, when imposing the initial condition  $u(x, 0) = \phi(x)$  for the corresponding moments as given in (4.15), then infinitely many and functionally independent invariant (equivalence) transformations can be constructed which all are compatible with this arbitrary but specifically chosen initial condition. Because, since e.g. the infinitesimals  $\xi = \xi(t, u_0, u_1)$ ,  $\eta_{u_0}(t, u_0, u_1)$  and  $\eta_{u_1}(t, u_0, u_1)$  can be chosen arbitrarily, one only has to guarantee that each of the initial conditions  $u_n(t = 0) = u_n(0)$ , for all  $n \geq 0$ , gets invariantly mapped into itself. This is achieved by demanding all infinitesimals to satisfy the restrictions

$$\left. \begin{aligned} \xi(t, u_0, u_1) \Big|_{\{t=0; u_n=u_n(0), \forall n \geq 0\}} &= 0, \\ \eta_{u_0}(t, u_0, u_1) \Big|_{\{t=0; u_n=u_n(0), \forall n \geq 0\}} &= 0, \quad \eta_{u_1}(t, u_0, u_1) \Big|_{\{t=0; u_n=u_n(0), \forall n \geq 0\}} = 0, \end{aligned} \right\} \quad (4.25)$$

$$\left. \begin{aligned} \eta_{u_{2n}}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3, \dots, u_{2n}, u_{2n+1}) \Big|_{\{t=0; u_n=u_n(0), \forall n \geq 0\}} &= 0, \quad n \geq 1, \\ \eta_{u_{2n+1}}(\xi, \eta_{u_0}, \eta_{u_1}, u_2, u_3, \dots, u_{2n}, u_{2n+1}) \Big|_{\{t=0; u_n=u_n(0), \forall n \geq 0\}} &= 0, \quad n \geq 1, \end{aligned} \right\} \quad (4.26)$$

where only the three infinitesimals  $\xi$ ,  $\eta_{u_0}$  and  $\eta_{u_1}$  can be chosen freely, while the remaining infinitesimals  $\eta_{2n}$  and  $\eta_{2n+1}$ , for all  $n \geq 1$ , are predetermined differential functions of their indicated arguments. The conditions (4.25), in accordance with (4.26), can be easily fulfilled e.g. by restricting the three arbitrary functions  $\xi$ ,  $\eta_{u_0}$  and  $\eta_{u_1}$  to

$$\left. \begin{aligned} \xi(t, u_0, u_1) &= f_0(t, u_0, u_1) \cdot e^{-\frac{\gamma_f^2}{t^2}}, \\ \eta_{u_0}(t, u_0, u_1) &= g_0(t, u_0, u_1) \cdot e^{-\frac{\gamma_g^2}{(u_0 - u_0(0))^2}}, \quad \eta_{u_1}(t, u_0, u_1) = h_0(t, u_0, u_1) \cdot e^{-\frac{\gamma_h^2}{(u_1 - u_1(0))^2}}, \end{aligned} \right\} \quad (4.27)$$

where  $f_0$ ,  $g_0$  and  $h_0$  are again (new) arbitrary functions, however, now restricted to the class of functions which are increasing slower than  $e^{1/r^2}$  at  $r = 0$ , where

$$r = \sqrt{t^2/\gamma_f^2 + (u_0 - u_0(0))^2/\gamma_g^2 + (u_1 - u_1(0))^2/\gamma_h^2}. \quad (4.28)$$

And, since in this case all differential functions  $\eta_{u_{2n}}$  and  $\eta_{u_{2n+1}}$ ,  $\forall n \geq 1$  have the special non-shifted affine property

$$\eta_{u_{2n}} \Big|_{\{\xi=0; \eta_{u_0}=0; \eta_{u_1}=0\}} = \eta_{u_{2n+1}} \Big|_{\{\xi=0; \eta_{u_0}=0; \eta_{u_1}=0\}} = 0, \quad \forall n \geq 1, \quad (4.29)$$

the conditions (4.26) all are automatically satisfied by the above restriction (4.27). Hence, an *infinite* set of functionally independent (non-privileged) invariant solutions for each moment  $u_n = u_n(t)$ ,  $\forall n \geq 0$  can be constructed from (4.27), which all satisfy the given initial condition  $u_n(t=0) = u_n(0)$  as imposed in (4.15).

That a priori no unique general solution, nor any unique invariant solution can be constructed, provides the reason that the PDE induced ODE system (4.18), although infinite in dimension, has to be treated as an unclosed system. It involves more unknown functions than there are determining equations, although *formally*, in a bijective manner, to each function within the hierarchy a corresponding equation can be mapped to. But, since the hierarchy (4.18) can be equivalently rewritten into the solved form (4.16) (or into (4.23), etc.) it explicitly reveals the fact that exactly two functions in this hierarchy, e.g.  $u_0(t)$  and  $u_1(t)$ , remain unknown, and without the precise knowledge of their global functional structure all remaining solutions  $u_n(t)$  for  $n \geq 2$  then remain unknown too. Even when posing sufficient initial conditions is not enough to yield a unique solution for the lower-level ODE system (4.15) as it is for the higher-level PDE equation (4.1). Without a prior modelling assumption on the ODE system (4.15), this system remains unclosed. Fortunately, the solutions of this particular case (4.11)-(4.14) possessed an analytical part in their functions for which the assumed Ansatz (4.19) expressed the correct functional behavior, though only in a very narrow and limited range. But, of course, for more general cases such a partial analytical structure is not always necessarily provided, and an Ansatz as (4.19) would then be misleading.<sup>†</sup>

Hence, the infinite forward recursively defined system of order two (4.18) does not possess a *unique* general solution. The degree of arbitrariness in having two unknown functions cannot be reduced, even when a sufficient number of initial conditions are imposed, simply due to the existing modus operandi (4.22) when constructing possible valid solutions. The same non-uniqueness problem we face for invariant solutions due to the construction principle (4.27), in that infinitely many functionally independent equivalence transformations can be constructed which all can be made compatible to any specifically chosen initial condition. Hence, the existence of an invariant solution for such a system is without any value, since it just represents an arbitrary solution among infinitely many other, equally privileged invariant solutions.

## 5. Conclusion

At the example of a lower-dimensional ODE-system this study has shown that an infinite and *forward* recursive hierarchy of differential equations carries all features of an unclosed system, and that, conclusively, all admitted invariance transformations may only be identified as equivalence transformations. To obtain from such systems an invariant solution which should show a certain particular functional structure is ultimately without value, since infinitely many functionally different and non-privileged invariant solutions can be constructed. In order to obtain valuable results, the infinite system needs to be closed by posing modelling assumptions which have to reflect the structure of the underlying (higher-level) equations from which the infinite system emerges.

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<sup>†</sup>Note that not only an analytical Ansatz as (4.19) would be misleading, but also any other, functionally different Ansatz can be misleading too, if no further external (exogenous) information is at hand. For example, choosing instead of an analytical a Fourier-expansion Ansatz would globally fail as the underlying functions (4.11)-(4.14) do not show any time-periodic behavior, at all.

Although in this study only an example of a lower-dimensional system has been considered, it is well to be expected that the very same non-uniqueness issues also occur in the higher-dimensional Friedmann-Keller hierarchy of moments for turbulent flows, and that it would be instructive to remember this example when trying to generate solutions or invariant solutions from the infinite and unmodelled Friedmann-Keller equations as its motivated e.g. in the work of Oberlack et al. Besides the problem of being defined forward recursively, the infinite Friedmann-Keller hierarchy faces another and further substantial problem, namely the singular problem of *not* showing a symmetric and thus complete coupling among all equations. As it was discussed at the end of Section 2, the problem that in each equation the occurring unknown higher order moment is not directly coupled to the corresponding next higher order equation, additionally even increases the degree of underdeterminedness of the Friedmann-Keller equations, which thus undoubtedly renders it altogether to an unclosed system.

Hence, the approach and the conclusions given in Oberlack & Rosteck (2010); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014); Waławczyk *et al.* (2014); Oberlack *et al.* (2014), about deriving physically relevant invariant solutions as so-called turbulent scaling-laws from first principles for the lower level moments within the infinite Friedmann-Keller hierarchy, is heavily misleading; all the more so as these first principle invariant solutions are additionally based on physically inconsistent symmetries (for more details, see Frewer *et al.* (2014a,b)). The general argument that these turbulent scaling laws are "clearly validated" in Oberlack & Rosteck (2011a,b); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014); Oberlack *et al.* (2014), by comparing to direct numerical simulations (DNS), is based on a fallacy (see Section 5 in Frewer *et al.* (2014a)). The problem is that this "validation" is always only performed for the lowest order moments, which, of course, can be robustly matched to the DNS data since there are enough free group parameters available to be fitted; in the very same sense as John von Neumann famously remarked many years ago (as quoted by Enrico Fermi to Freeman Dyson, see e.g. Dyson (2004)):

*"With four parameters I can fit an elephant, and with five I can make him wiggle his trunk."*

But, as soon as the higher order correlations functions get fitted, the curve-fitting procedure of Oberlack et al. consistently fails. This outcome is supported by a mathematical proof given in Frewer *et al.* (2014a) (see Appendix D), which clearly shows that the Lie-group based turbulent scaling laws for all higher order velocity correlations as derived in Oberlack & Rosteck (2010); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014); Waławczyk *et al.* (2014); Oberlack *et al.* (2014) are *not* consistent to the scaling of the lowest order correlation function, which for shear flows is the mean velocity field. In other words, the proof in Frewer *et al.* (2014a) shows that for the lowest correlation order  $n = 1$  no contradiction exists, only as from  $n \geq 2$  onwards the contradiction starts, i.e. while the mean velocity field can be robustly matched to the DNS data, it consistently fails for all higher order correlation functions and gets more pronounced the higher the correlation order  $n$  is.

Hence, by always only matching or comparing the DNS data to the lowest order velocity correlation of a specific flow, as it was done particularly in Oberlack & Rosteck (2011a,b); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014); Oberlack *et al.* (2014), i.e. by matching or comparing only to the mean velocity profile for wall-bounded (shear) flows and only to the two-point velocity correlation for isotropic flows, is definitely not enough to be a true validation of the Lie-group based scaling theory. In particular, as this theory by Oberlack et al. which specifically considers the infinite (unclosed) system of all multi-point correlation equations and which thus is designed and laid-out to be a scaling theory for all higher order velocity correlations, special attention has to be devoted to the prediction value of all those correlation functions which go beyond the lowest order. And exactly this has been investigated in Frewer *et al.* (2014a) (Section 5), which then gives a completely different picture than the "validation"

procedure in Oberlack & Rosteck (2011*a,b*); Oberlack & Zieleniewicz (2013); Avsarkisov *et al.* (2014); Oberlack *et al.* (2014) is trying to suggest.

Taken together, the key reason for this failure is twofold: i) Oberlack *et al.* considers the infinite and forward recursive Friedmann-Keller hierarchy of moments as a closed system, which, as was clarified in this study at an analogous lower-dimensional system of ODEs, is not the case; ii) Oberlack *et al.* considers all layers of statistical description of turbulence endogenously without involving at the same time the underlying deterministic Navier-Stokes system — a methodological approach which obviously is incomplete and even incorrect, because it are the deterministic equations which due to their spatially nonlocal and temporally chaotic behavior induce the statistical equations and not vice versa. In other words, in order to obtain valuable and physically consistent results within a statistical description of the physically quantifiable world, it is necessary to include all underlying processes up to the fluctuating level into this description and not to ignore them if they exist.

### A. Alternative form of a transverse vector field

The transverse component  $\mathbf{A}^\perp(\mathbf{x})$  of an arbitrary vector field  $\mathbf{A}(\mathbf{x})$ , when decomposed into a longitudinal and transverse component  $\mathbf{A}(\mathbf{x}) = \mathbf{A}^\parallel(\mathbf{x}) + \mathbf{A}^\perp(\mathbf{x})$  respectively, and, which decays sufficiently fast for  $|\mathbf{x}| \rightarrow \infty$ , is defined as (see e.g. Stewart (2008))

$$\mathbf{A}^\perp(\mathbf{x}) := \mathbf{A}(\mathbf{x}) + \nabla \int \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad (\text{A.1})$$

confirming its defining (solenoidal) property  $\nabla \cdot \mathbf{A}^\perp(\mathbf{x}) = 0$ , when making use of the standard identity

$$\delta^3(\mathbf{x} - \mathbf{x}') = -\Delta \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (\text{A.2})$$

Relation (A.1) can be written in an alternative form, by rewriting it as

$$\begin{aligned} \mathbf{A}^\perp(\mathbf{x}) &= \mathbf{A}(\mathbf{x}) + \nabla \int \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \\ &= \mathbf{A}(\mathbf{x}) + \nabla \int \nabla' \cdot \left( \frac{\mathbf{A}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' - \nabla \int \mathbf{A}(\mathbf{x}') \cdot \left( \nabla' \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' \\ &= \mathbf{A}(\mathbf{x}) - \nabla \int \mathbf{A}(\mathbf{x}') \cdot \left( \nabla' \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' \\ &= \mathbf{A}(\mathbf{x}) + \nabla \int \mathbf{A}(\mathbf{x}') \cdot \left( \nabla \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' \\ &= \mathbf{A}(\mathbf{x}) + \int \mathbf{A}(\mathbf{x}') \cdot \left( \nabla \otimes \nabla \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) d^3\mathbf{x}' \\ &= \int d^3\mathbf{x}' \left( \delta^3(\mathbf{x} - \mathbf{x}') \cdot \mathbf{1} + \nabla \otimes \nabla \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) \cdot \mathbf{A}(\mathbf{x}'), \end{aligned} \quad (\text{A.3})$$

where we made use of Gauss' theorem for the second term in the second line, which resulted into a vanishing surface integral due vanishing contributions of the vector field  $\mathbf{A}(\mathbf{x})$  at infinity. Hence, the alternative relation (A.3) for a transverse vector field allows to define an operator  $\mathcal{P}$  of orthoprojection to solenoidal vector fields:  $\mathbf{A} \mapsto \mathcal{P}[\mathbf{A}] = \mathbf{A}^\perp$ . The explicit expression for  $\mathcal{P}$  acting on any arbitrary vector-field  $\mathbf{A}$  vanishing sufficiently fast at infinity then has the form given by (A.3) (see e.g. also Appendix B in Frewer *et al.* (2014*a*)):

$$\mathcal{P}[\mathbf{A}] = \int d^3\mathbf{x}' \rho(\mathbf{x} - \mathbf{x}') \cdot \mathbf{A}(\mathbf{x}'), \quad (\text{A.4})$$

with the kernel

$$\boldsymbol{\rho}(\mathbf{x} - \mathbf{x}') := \delta^3(\mathbf{x} - \mathbf{x}') \cdot \mathbf{1} + \nabla \otimes \nabla \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}. \quad (\text{A.5})$$

By construction, the projection properties of  $\mathcal{P}$  are

$$\left. \begin{aligned} \mathcal{P}[\mathbf{A}] &= \mathbf{A}, & \text{if } \nabla \cdot \mathbf{A} &= 0, \\ \mathcal{P}[\mathbf{A}] &= \mathbf{0}, & \text{if } \nabla \times \mathbf{A} &= \mathbf{0}, \text{ e.g. if } \mathbf{A} = \nabla\varphi. \end{aligned} \right\} \quad (\text{A.6})$$

## B. The Hopf equation in two different representations

If we assume that  $\nabla \cdot \mathbf{y} \neq 0$ , for any arbitrary vector field  $\mathbf{y} = \mathbf{y}(\mathbf{x})$  which is decaying sufficiently fast at infinity, the functional Hopf equation as given by (2.1) can be represented in two alternative ways. Either in a representation relative to the full vector field  $\mathbf{y}$ , or in a representation relative to its projected transverse (solenoidal) part  $\mathbf{y}^\perp$ .

### B.1. The Hopf equation in the full-field representation

According to the defining relation (2.2), the functional Hopf equation (2.1) will take the form

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= \int \left( y_\alpha(\mathbf{x}) - \partial_\alpha \varphi(\mathbf{x}) \right) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ &= \int y_\alpha(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ &\quad + \int \varphi(\mathbf{x}) \left( i \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\partial}{\partial x_\alpha} \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ &= \int y_\alpha(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} + i \int \varphi(\mathbf{x}) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} d^3 \mathbf{x} \\ &= \int y_\alpha(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ &\quad + i \int y_\gamma(\mathbf{x}') \left( \frac{\partial}{\partial x'_\gamma} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} d^3 \mathbf{x}' d^3 \mathbf{x} \\ &= \int y_\alpha(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x}) \delta y_\beta(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} \right) d^3 \mathbf{x} \\ &\quad + i \int y_\alpha(\mathbf{x}) \left( \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) \frac{\partial^2}{\partial x'_\beta \partial x'_\gamma} \frac{\delta^2 \Phi}{\delta y_\beta(\mathbf{x}') \delta y_\gamma(\mathbf{x}')} d^3 \mathbf{x}' d^3 \mathbf{x}. \end{aligned} \quad (\text{B.1})$$

Note that in going from the second to the third equality, the incompressibility condition in functional space (Hopf, 1952, p. 97)<sup>†</sup> has been made use of

$$\frac{\partial}{\partial x_\alpha} \frac{\delta \Phi}{\delta y_\alpha(\mathbf{x})} = 0. \quad (\text{B.2})$$

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<sup>†</sup>Note that the incompressibility condition only refers to the velocity field  $\mathbf{u}$  of the flow, and *not* to the auxiliary field  $\mathbf{y}$ . In other words, although we assume  $\nabla \cdot \mathbf{y} \neq 0$ , i.e.  $\mathbf{y} \neq \mathbf{y}^\perp$ , we still have incompressibility in the velocity field  $\nabla \cdot \mathbf{u} = 0$ , i.e.  $\mathbf{u} = \mathbf{u}^\perp$ ; and in functional space it is expressed as (B.2).



## B.2. The Hopf equation in the transverse representation

In order to transform the Hopf equation (2.1) into a representation where only the transverse fields  $\mathbf{y}^\perp$  appear, it is necessary to first transform the functional derivatives according to relation (2.2) in applying the corresponding functional chain rule

$$\frac{\delta}{\delta y_\alpha(\mathbf{x})} = \int d^3\mathbf{z} \frac{\delta y_\kappa^\perp(\mathbf{z})}{\delta y_\alpha(\mathbf{x})} \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})}, \quad (\text{B.3})$$

$$\frac{\delta^2}{\delta y_\alpha(\mathbf{x})\delta y_\beta(\mathbf{x})} = \int d^3\mathbf{z} \frac{\delta^2 y_\kappa^\perp(\mathbf{z})}{\delta y_\alpha(\mathbf{x})\delta y_\beta(\mathbf{x})} \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} + \int d^3\mathbf{z} d^3\mathbf{z}' \frac{\delta y_\kappa^\perp(\mathbf{z})}{\delta y_\alpha(\mathbf{x})} \frac{\delta y_\lambda^\perp(\mathbf{z}')}{\delta y_\beta(\mathbf{x})} \frac{\delta^2}{\delta y_\kappa^\perp(\mathbf{z})\delta y_\lambda^\perp(\mathbf{z}')}. \quad (\text{B.4})$$

Let's first consider the transformation of the single functional derivative in the viscous term:

$$\begin{aligned} \frac{\delta}{\delta y_\alpha(\mathbf{x})} &= \int d^3\mathbf{z} \left( \delta_{\alpha\kappa} \delta^3(\mathbf{z} - \mathbf{x}) + \frac{\partial}{\partial z_\kappa} \int \frac{\partial'_\alpha \delta^3(\mathbf{x}' - \mathbf{x})}{4\pi|\mathbf{z} - \mathbf{x}'|} d^3\mathbf{x}' \right) \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} \\ &= \frac{\delta}{\delta y_\alpha^\perp(\mathbf{x})} - \int d^3\mathbf{z} \left( \frac{\partial}{\partial z_\kappa} \int \delta^3(\mathbf{x}' - \mathbf{x}) \frac{\partial}{\partial x'_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}'|} d^3\mathbf{x}' \right) \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} \\ &= \frac{\delta}{\delta y_\alpha^\perp(\mathbf{x})} - \int d^3\mathbf{z} \left( \frac{\partial}{\partial z_\kappa} \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}|} \right) \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} \\ &= \frac{\delta}{\delta y_\alpha^\perp(\mathbf{x})} + \int d^3\mathbf{z} \left( \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}|} \right) \frac{\partial}{\partial z_\kappa} \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})}. \end{aligned} \quad (\text{B.5})$$

If this operator now acts on the characteristic functional  $\Phi$ , which itself transforms invariantly (Hopf, 1952, p. 93)

$$\Phi[\mathbf{y}(\mathbf{x}), t] = \Phi[\mathbf{y}^\perp(\mathbf{x}), t], \quad (\text{B.6})$$

then the second right-hand-side term in the last line of (B.5) results to zero due to the incompressibility condition (B.2). Hence we obtain the following invariant result

$$\frac{\delta \Phi[\mathbf{y}(\mathbf{x}), t]}{\delta y_\alpha(\mathbf{x}')} = \frac{\delta \Phi[\mathbf{y}^\perp(\mathbf{x}), t]}{\delta y_\alpha^\perp(\mathbf{x}')}. \quad (\text{B.7})$$

Similar for the second derivative in the inertial term, which also gives the invariant result<sup>†</sup>

$$\begin{aligned} \frac{\delta^2 \Phi}{\delta y_\alpha(\mathbf{x})\delta y_\beta(\mathbf{x})} &= \int d^3\mathbf{z} d^3\mathbf{z}' \left( \delta_{\alpha\kappa} \delta^3(\mathbf{z} - \mathbf{x}) + \frac{\partial}{\partial z_\kappa} \int \frac{\partial'_\alpha \delta^3(\mathbf{x}' - \mathbf{x})}{4\pi|\mathbf{z} - \mathbf{x}'|} d^3\mathbf{x}' \right) \\ &\quad \cdot \left( \delta_{\beta\lambda} \delta^3(\mathbf{z}' - \mathbf{x}) + \frac{\partial}{\partial z'_\lambda} \int \frac{\partial''_\beta \delta^3(\mathbf{x}'' - \mathbf{x})}{4\pi|\mathbf{z}' - \mathbf{x}''|} d^3\mathbf{x}'' \right) \frac{\delta^2 \Phi}{\delta y_\kappa^\perp(\mathbf{z})\delta y_\lambda^\perp(\mathbf{z}')} \\ &= \int d^3\mathbf{z} d^3\mathbf{z}' \left( \delta_{\alpha\kappa} \delta^3(\mathbf{z} - \mathbf{x}) - \frac{\partial}{\partial z_\kappa} \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}|} \right) \\ &\quad \cdot \left( \delta_{\beta\lambda} \delta^3(\mathbf{z}' - \mathbf{x}) - \frac{\partial}{\partial z'_\lambda} \frac{\partial}{\partial x_\beta} \frac{1}{4\pi|\mathbf{z}' - \mathbf{x}|} \right) \frac{\delta^2 \Phi}{\delta y_\kappa^\perp(\mathbf{z})\delta y_\lambda^\perp(\mathbf{z}')} \end{aligned}$$

<sup>†</sup>Note that since the functional relation (2.2) for  $\mathbf{y}^\perp$  is linear in  $\mathbf{y}$ , the first term of the chain rule (B.4) gives no contribution.

$$\begin{aligned}
&= \frac{\delta^2 \Phi}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} \\
&\quad + \int d^3 \mathbf{z} d^3 \mathbf{z}' \delta_{\alpha\kappa} \delta^3(\mathbf{z} - \mathbf{x}) \frac{\partial}{\partial x_\beta} \frac{1}{4\pi|\mathbf{z}' - \mathbf{x}|} \cdot \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} \left( \frac{\partial}{\partial z'_\lambda} \frac{\delta \Phi}{\delta y_\lambda^\perp(\mathbf{z}')} \right) \\
&\quad + \int d^3 \mathbf{z} d^3 \mathbf{z}' \delta_{\beta\lambda} \delta^3(\mathbf{z}' - \mathbf{x}) \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}|} \cdot \frac{\delta}{\delta y_\lambda^\perp(\mathbf{z}')} \left( \frac{\partial}{\partial z_\kappa} \frac{\delta \Phi}{\delta y_\kappa^\perp(\mathbf{z})} \right) \\
&\quad + \int d^3 \mathbf{z} d^3 \mathbf{z}' \left( \frac{\partial}{\partial x_\alpha} \frac{1}{4\pi|\mathbf{z} - \mathbf{x}|} \right) \left( \frac{\partial}{\partial x_\beta} \frac{1}{4\pi|\mathbf{z}' - \mathbf{x}|} \right) \cdot \frac{\partial}{\partial z_\kappa} \frac{\delta}{\delta y_\kappa^\perp(\mathbf{z})} \left( \frac{\partial}{\partial z'_\lambda} \frac{\delta \Phi}{\delta y_\lambda^\perp(\mathbf{z}')} \right) \\
&= \frac{\delta^2 \Phi}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})}, \tag{B.8}
\end{aligned}$$

where again we made use of the incompressibility condition (B.2). Now, since the characteristic functional  $\Phi$  transforms invariantly (B.6), the transformed Hopf equation (2.1) in the transverse representation finally takes the form

$$\frac{\partial \Phi}{\partial t} = \int y_\alpha^\perp(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} + \nu \Delta \frac{\delta \Phi}{\delta y_\alpha^\perp(\mathbf{x})} \right) d^3 \mathbf{x}. \tag{B.9}$$

### C. The MPC equations in two different representations

Similar as in the case for the Hopf equation (see Appendix B), the MPC equations for the incompressible Navier-Stokes velocity field  $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$  can be represented in two alternative ways too. Again, either in a representation relative to the full MPC fields<sup>†</sup>

$$H_{\alpha_{\{n\}}} := H_{\alpha_1 \dots \alpha_n} := \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle, \tag{C.1}$$

or in a representation relative to the transversely projected MPC fields

$$H_{\alpha_{\{n\}}}^\perp := H_{\alpha_1 \dots \alpha_n}^\perp := \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle^\perp. \tag{C.2}$$

Note that due to (2.10), along with its evaluation (2.11), we have the obvious invariant relation  $H_{\alpha_{\{n\}}}^\perp = H_{\alpha_{\{n\}}}$ . This invariance can be readily verified independently from relation (2.10), by recognizing that i) the (fluctuating) velocity field is solenoidal  $\partial_{\alpha_k} u_{\alpha_k}(\mathbf{x}_k) = 0$ , for any index  $k$ , i.e.  $u_{\alpha_k}(\mathbf{x}_k) = u_{\alpha_k}^\perp(\mathbf{x}_k)$ , and ii) that any usual (non-functional) differential operator acting locally on any point  $\mathbf{x}_k$  is commuting with the averaging (ensemble) operator  $\langle \cdot \rangle$ . Hence we have the relation

$$\partial_{\alpha_k} \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_k}(\mathbf{x}_k) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle = \langle u_{\alpha_1}(\mathbf{x}_1) \cdots \partial_{\alpha_k} u_{\alpha_k}(\mathbf{x}_k) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle = 0 \tag{C.3}$$

$$\Longleftrightarrow$$

$$\langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle = \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle^\perp. \tag{C.4}$$

But careful: When taking the smooth and regular limit of a zero-correlation length between any two points  $|\mathbf{x}_k - \mathbf{x}_l| \rightarrow 0$ , the invariance (C.4) no longer holds, because obviously

$$\begin{aligned}
&\partial_{\alpha_k} \lim_{\mathbf{x}_l \rightarrow \mathbf{x}_k} \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_k}(\mathbf{x}_k) \cdots u_{\alpha_l}(\mathbf{x}_l) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle \\
&= \langle u_{\alpha_1}(\mathbf{x}_1) \cdots u_{\alpha_k}(\mathbf{x}_k) \cdots \partial_{\alpha_k} u_{\alpha_l}(\mathbf{x}_k) \cdots u_{\alpha_n}(\mathbf{x}_n) \rangle \neq 0. \tag{C.5}
\end{aligned}$$

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<sup>†</sup>The  $H$ -notation for the (instantaneous) MPC functions in (C.1) is taken from Oberlack & Rosteck (2010); see also Frewer *et al.* (2014a) for a critical discussion on using this notation, in particular for invariance analysis.

## C.1. The MPC equations in the transverse representation

The underlying MPC-generating equation is given by (2.12)

$$\frac{\partial \Phi^n}{\partial t} = \int y_\alpha^\perp(\mathbf{x}) \left( i \frac{\partial}{\partial x_\beta} \frac{\delta^2 \Phi^{n+1}}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} + \nu \Delta \frac{\delta \Phi^n}{\delta y_\alpha^\perp(\mathbf{x})} \right) d^3 \mathbf{x}, \quad \forall n \geq 1, \quad (\text{C.6})$$

where the hierarchy of functionals  $\Phi^n$  is given by (2.9), along with (2.10) and (2.11). Three terms are contributing to equation (C.6): the temporal term on the left-hand side and the inertial and viscous terms on the right-hand side, which for each order  $n$  evaluates to (shown here only up third order)

$$n = 1 : \quad \frac{\partial \Phi^1}{\partial t} = \int d^3 \mathbf{x}_1 y_{\alpha_1}^\perp(\mathbf{x}_1) \left( i \frac{\partial}{\partial t} \langle u_{\alpha_1}(\mathbf{x}_1) \rangle^\perp \right), \quad (\text{C.7})$$

$$\frac{\delta^2 \Phi^2}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} = -\frac{2 \cdot 1}{2!} \langle u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) \rangle^\perp, \quad (\text{C.8})$$

$$\frac{\delta \Phi^1}{\delta y_\alpha^\perp(\mathbf{x})} = i \langle u_\alpha(\mathbf{x}) \rangle^\perp, \quad (\text{C.9})$$

$$n = 2 : \quad \frac{\partial \Phi^2}{\partial t} = \int d^3 \mathbf{x}_1 d^3 \mathbf{x}_2 y_{\alpha_1}^\perp(\mathbf{x}_1) y_{\alpha_2}^\perp(\mathbf{x}_2) \left( -\frac{1}{2!} \frac{\partial}{\partial t} \langle u_{\alpha_1}(\mathbf{x}_1) u_{\alpha_2}(\mathbf{x}_2) \rangle^\perp \right), \quad (\text{C.10})$$

$$\frac{\delta^2 \Phi^3}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} = \int d^3 \mathbf{x}_3 y_{\alpha_3}^\perp(\mathbf{x}_3) \left( -i \frac{3 \cdot 2}{3!} \langle u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) u_{\alpha_3}(\mathbf{x}_3) \rangle^\perp \right), \quad (\text{C.11})$$

$$\frac{\delta \Phi^2}{\delta y_\alpha^\perp(\mathbf{x})} = \int d^3 \mathbf{x}_2 y_{\alpha_2}^\perp(\mathbf{x}_2) \left( -\frac{2}{2!} \langle u_\alpha(\mathbf{x}) u_{\alpha_2}(\mathbf{x}_2) \rangle^\perp \right), \quad (\text{C.12})$$

$$n = 3 : \quad \frac{\partial \Phi^3}{\partial t} = \int d^3 \mathbf{x}_1 d^3 \mathbf{x}_2 d^3 \mathbf{x}_3 y_{\alpha_1}^\perp(\mathbf{x}_1) y_{\alpha_2}^\perp(\mathbf{x}_2) y_{\alpha_3}^\perp(\mathbf{x}_3) \cdot \left( -i \frac{1}{3!} \frac{\partial}{\partial t} \langle u_{\alpha_1}(\mathbf{x}_1) u_{\alpha_2}(\mathbf{x}_2) u_{\alpha_3}(\mathbf{x}_3) \rangle^\perp \right), \quad (\text{C.13})$$

$$\frac{\delta^2 \Phi^4}{\delta y_\alpha^\perp(\mathbf{x}) \delta y_\beta^\perp(\mathbf{x})} = \int d^3 \mathbf{x}_3 d^3 \mathbf{x}_4 y_{\alpha_3}^\perp(\mathbf{x}_3) y_{\alpha_4}^\perp(\mathbf{x}_4) \cdot \left( \frac{4 \cdot 3}{4!} \langle u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) u_{\alpha_3}(\mathbf{x}_3) u_{\alpha_4}(\mathbf{x}_4) \rangle^\perp \right), \quad (\text{C.14})$$

$$\frac{\delta \Phi^3}{\delta y_\alpha^\perp(\mathbf{x})} = \int d^3 \mathbf{x}_2 d^3 \mathbf{x}_3 y_{\alpha_2}^\perp(\mathbf{x}_2) y_{\alpha_3}^\perp(\mathbf{x}_3) \left( -i \frac{3}{3!} \langle u_\alpha(\mathbf{x}) u_{\alpha_2}(\mathbf{x}_2) u_{\alpha_3}(\mathbf{x}_3) \rangle^\perp \right), \quad (\text{C.15})$$

and so on for all higher orders. When inserting these results for each order into equation (C.6), two things should be noted or to be carried out before dropping the integrals in order to obtain the MPC equations in the transverse representation: i) The integrals for the inertial and viscous terms must be fully symmetrized in their integration variables in accord with the integrals from the temporal term, e.g. consider the second order result for the inertial term (C.11), which,

through equation (C.6) and in accord with result (C.10), has to be symmetrized to

$$\begin{aligned}
& \int d^3 \mathbf{x} d^3 \mathbf{x}_3 y_\alpha^\perp(\mathbf{x}) y_{\alpha_3}^\perp(\mathbf{x}_3) \frac{\partial}{\partial x_\beta} \langle u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) u_{\alpha_3}(\mathbf{x}_3) \rangle^\perp \\
&= \int d^3 \mathbf{x}_1 d^3 \mathbf{x}_2 y_{\alpha_1}^\perp(\mathbf{x}_1) y_{\alpha_2}^\perp(\mathbf{x}_2) \left( \frac{1}{2} \frac{\partial}{\partial x_{1,\sigma_1}} \langle u_{\alpha_1}(\mathbf{x}_1) u_{\alpha_2}(\mathbf{x}_2) u_{\sigma_1}(\mathbf{x}_1) \rangle^\perp \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial}{\partial x_{2,\sigma_2}} \langle u_{\alpha_1}(\mathbf{x}_1) u_{\alpha_2}(\mathbf{x}_2) u_{\sigma_2}(\mathbf{x}_2) \rangle^\perp \right) \\
&= \int d^3 \mathbf{x}_1 d^3 \mathbf{x}_2 y_{\alpha_1}^\perp(\mathbf{x}_1) y_{\alpha_2}^\perp(\mathbf{x}_2) \left( \frac{1}{2} \frac{\partial}{\partial x_{1,\sigma_1}} \hat{H}_{\alpha_1 \alpha_2 \sigma_1}^\perp + \frac{1}{2} \frac{\partial}{\partial x_{2,\sigma_2}} \hat{H}_{\alpha_1 \alpha_2 \sigma_2}^\perp \right), \quad (\text{C.16})
\end{aligned}$$

and ii) the invariance condition  $\mathbf{H}_n^\perp = \mathbf{H}_n$  (C.4) is only valid if the MPC function of  $n$ -th order  $\mathbf{H}_n^\perp = (H_{\alpha_1 \dots \alpha_n}^\perp)$  is also evaluated at  $n$  distinct points  $\mathbf{x}_n$ . As soon as it's evaluated in less than  $n$  distinct points, the invariance no longer holds (C.5). These lower-dimensional MPC functions will be denoted by  $\hat{\mathbf{H}}_n^\perp$ , as it was already done in (C.16), in order to avoid any uncertainties in the notation. For them, as already said, the invariance condition does not hold, i.e.  $\hat{\mathbf{H}}_n^\perp \neq \hat{\mathbf{H}}_n$ . Following both these instructions i) and ii), we finally obtain the hierarchy of MPC equations in the transverse representation (shown here only up third order)

$$n = 1 : \quad \frac{\partial H_{\alpha_1}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \sigma_1}^\perp}{\partial x_{1,\sigma_1}} - \nu \frac{\partial^2 H_{\alpha_1}}{\partial x_{1,\sigma_1} \partial x_{1,\sigma_1}} = 0, \quad (\text{C.17})$$

$$n = 2 : \quad \frac{\partial H_{\alpha_1 \alpha_2}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}^\perp}{\partial x_{1,\sigma_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}^\perp}{\partial x_{2,\sigma_2}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2}}{\partial x_{1,\sigma_1} \partial x_{1,\sigma_1}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2}}{\partial x_{2,\sigma_2} \partial x_{2,\sigma_2}} = 0, \quad (\text{C.18})$$

$$\begin{aligned}
n = 3 : \quad & \frac{\partial H_{\alpha_1 \alpha_2 \alpha_3}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_1}^\perp}{\partial x_{1,\sigma_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_2}^\perp}{\partial x_{2,\sigma_2}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_3}^\perp}{\partial x_{3,\sigma_3}} \\
& - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{1,\sigma_1} \partial x_{1,\sigma_1}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{2,\sigma_2} \partial x_{2,\sigma_2}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{3,\sigma_3} \partial x_{3,\sigma_3}} = 0, \quad (\text{C.19})
\end{aligned}$$

where the coupling within the hierarchy is given through the definition

$$\hat{H}_{\alpha_1 \dots \alpha_n \sigma_k}^\perp = \left( \lim_{\mathbf{x}_{n+1} \rightarrow \mathbf{x}_k; \alpha_{n+1} \rightarrow \sigma_k} H_{\alpha_1 \dots \alpha_n \alpha_{n+1}} \right)^\perp, \quad \forall k \text{ between } 1 \leq k \leq n. \quad (\text{C.20})$$

Note that the infinite hierarchy of MPC equations is accompanied by the following obvious incompressibility constraints

$$\frac{\partial H_{\alpha_1 \dots \alpha_n}}{\partial x_{k,\alpha_k}} = 0, \quad \forall k \text{ between } 1 \leq k \leq n, \quad (\text{C.21})$$

$$\frac{\partial \hat{H}_{\alpha_1 \dots \alpha_n \alpha_l}^\perp}{\partial x_{k,\alpha_k}} = 0, \quad \forall k, l \text{ between } 1 \leq (k, l) \leq n, \text{ for } k \neq l. \quad (\text{C.22})$$

## C.2. The MPC equations in the full-field representation

The full-field representation of the MPC equations follows directly from (C.17)-(C.19) by decomposing each of the inertial transverse terms according to relation (2.2). But, as written, relation (2.2) only applies for vector fields. Therefore it is necessary to first extend this definition to also include tensor fields of arbitrary rank such that it applies to (C.17)-(C.19).

Following the principle of (2.2) it can straightforwardly be generalized to construct transverse (solenoidal) tensor fields

$$A_{\alpha_1 \dots \alpha_n}^\perp(\mathbf{x}) = A_{\alpha_1 \dots \alpha_n}(\mathbf{x}) + \frac{1}{n} \sum_{k=1}^n \partial_{\alpha_k} \int \frac{\partial'_{\alpha'_k} A_{\alpha_1 \dots \alpha'_k \dots \alpha_n}(\mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \quad (\text{C.23})$$

where  $\mathbf{A}_n = (A_{\alpha_1 \dots \alpha_n})$  is an arbitrary tensor field of rank  $n$  decaying sufficiently fast at infinity. By construction, (C.23) is in accord with (2.2) since its full-ranked divergence vanishes<sup>†</sup>

$$\partial_{\alpha_1 \dots \alpha_n} A_{\alpha_1 \dots \alpha_n}^\perp(\mathbf{x}) = 0. \quad (\text{C.24})$$

It is obvious that relation (C.23) also further extends to *multi*-point tensor fields, hence we obtain the following decompositions of the transverse MPC fields appearing in (C.17)-(C.19) (shown here only up to second order)

$$\begin{aligned} n=1: \quad \frac{\partial \hat{H}_{\alpha_1 \sigma_1}^\perp(\mathbf{x}_1)}{\partial x_{1, \sigma_1}} &= \frac{\partial \hat{H}_{\alpha_1 \sigma_1}(\mathbf{x}_1)}{\partial x_{1, \sigma_1}} + \frac{\partial}{\partial x_{1, \alpha_1}} \int \frac{d^3\mathbf{x}'_1}{4\pi|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\partial^2 \hat{H}_{\alpha'_1 \sigma'_1}(\mathbf{x}'_1)}{\partial x'_{1, \alpha'_1} \partial x'_{1, \sigma'_1}} \\ &= \frac{\partial \hat{H}_{\alpha_1 \sigma_1}(\mathbf{x}_1)}{\partial x_{1, \sigma_1}} + \frac{\partial}{\partial x_{1, \alpha_1}} \int \frac{d^3\mathbf{x}'_1}{4\pi|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\partial^2 \langle u_{\alpha'_1}(x'_1) u_{\sigma'_1}(x'_1) \rangle}{\partial x'_{1, \alpha'_1} \partial x'_{1, \sigma'_1}} \\ &= \frac{\partial \hat{H}_{\alpha_1 \sigma_1}(\mathbf{x}_1)}{\partial x_{1, \sigma_1}} + \frac{\partial \langle p(\mathbf{x}_1) \rangle}{\partial x_{1, \alpha_1}} =: \frac{\partial \hat{H}_{\alpha_1 \sigma_1}(\mathbf{x}_1)}{\partial x_{1, \sigma_1}} + \frac{\partial I_{[1]}(\mathbf{x}_1)}{\partial x_{1, \alpha_1}}, \end{aligned} \quad (\text{C.25})$$

$$\begin{aligned} n=2: \quad & \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}^\perp(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{1, \sigma_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}^\perp(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{2, \sigma_2}} \\ &= \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{1, \sigma_1}} + \frac{1}{2} \frac{\partial}{\partial x_{1, \alpha_1}} \int \frac{d^3\mathbf{x}'_1}{4\pi|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\partial^2 \hat{H}_{\alpha'_1 \alpha_2 \sigma'_1}(\mathbf{x}'_1, \mathbf{x}_2)}{\partial x'_{1, \alpha'_1} \partial x'_{1, \sigma'_1}} \\ & \quad + \frac{1}{2} \frac{\partial}{\partial x_{2, \alpha_2}} \int \frac{d^3\mathbf{x}'_2}{4\pi|\mathbf{x}_2 - \mathbf{x}'_2|} \frac{\partial^2 \hat{H}_{\alpha_1 \alpha'_2 \sigma'_2}(\mathbf{x}_1, \mathbf{x}'_2)}{\partial x'_{2, \alpha'_2} \partial x'_{2, \sigma'_2}} \\ & \quad + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{2, \sigma_2}} + \frac{1}{2} \frac{\partial}{\partial x_{2, \alpha_2}} \int \frac{d^3\mathbf{x}'_2}{4\pi|\mathbf{x}_2 - \mathbf{x}'_2|} \frac{\partial^2 \hat{H}_{\alpha_1 \alpha'_2 \sigma'_2}(\mathbf{x}_1, \mathbf{x}'_2)}{\partial x'_{2, \alpha'_2} \partial x'_{2, \sigma'_2}} \\ & \quad + \frac{1}{2} \frac{\partial}{\partial x_{1, \alpha_1}} \int \frac{d^3\mathbf{x}'_1}{4\pi|\mathbf{x}_1 - \mathbf{x}'_1|} \frac{\partial^2 \hat{H}_{\alpha'_1 \alpha_2 \sigma'_1}(\mathbf{x}'_1, \mathbf{x}_2)}{\partial x'_{1, \alpha'_1} \partial x'_{1, \sigma'_1}} \\ &= \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{1, \sigma_1}} + \frac{\partial \langle p(\mathbf{x}_1) u_{\alpha_2}(\mathbf{x}_2) \rangle}{\partial x_{1, \alpha_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{2, \sigma_2}} + \frac{\partial \langle u_{\alpha_1}(\mathbf{x}_1) p(\mathbf{x}_2) \rangle}{\partial x_{2, \alpha_2}} \\ &=: \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{1, \sigma_1}} + \frac{\partial I_{\alpha_2[1]}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{1, \alpha_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{2, \sigma_2}} + \frac{\partial I_{\alpha_1[2]}(\mathbf{x}_1, \mathbf{x}_2)}{\partial x_{2, \alpha_2}}, \end{aligned} \quad (\text{C.26})$$

<sup>†</sup>Note that only a full ranked divergence on  $A_{\alpha_1 \dots \alpha_n}^\perp$  (C.23) vanishes, i.e. when all indices are contracted. A partial divergence where not all indices are contracted does not result to zero. Hence, the partial divergences of the transverse MPC fields appearing in (C.17)-(C.19) are non-zero.

wich then according to (C.17)-(C.19) finally results into the infinite hierarchy of MPC equations in the full-field representation (shown here only up to third order)

$$n = 1 : \quad \frac{\partial H_{\alpha_1}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \sigma_1}}{\partial x_{1, \sigma_1}} + \frac{\partial I_{[1]}}{\partial x_{1, \alpha_1}} - \nu \frac{\partial^2 H_{\alpha_1}}{\partial x_{1, \sigma_1} \partial x_{1, \sigma_1}} = 0, \quad (\text{C.27})$$

$$n = 2 : \quad \frac{\partial H_{\alpha_1 \alpha_2}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_1}}{\partial x_{1, \sigma_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \sigma_2}}{\partial x_{2, \sigma_2}} + \frac{\partial I_{\alpha_2 [1]}}{\partial x_{1, \alpha_1}} + \frac{\partial I_{\alpha_1 [2]}}{\partial x_{2, \alpha_2}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2}}{\partial x_{1, \sigma_1} \partial x_{1, \sigma_1}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2}}{\partial x_{2, \sigma_2} \partial x_{2, \sigma_2}} = 0, \quad (\text{C.28})$$

$$n = 3 : \quad \frac{\partial H_{\alpha_1 \alpha_2 \alpha_3}}{\partial t} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_1}}{\partial x_{1, \sigma_1}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_2}}{\partial x_{2, \sigma_2}} + \frac{\partial \hat{H}_{\alpha_1 \alpha_2 \alpha_3 \sigma_3}}{\partial x_{3, \sigma_3}} + \frac{\partial I_{\alpha_2 \alpha_3 [1]}}{\partial x_{1, \alpha_1}} + \frac{\partial I_{\alpha_1 \alpha_3 [2]}}{\partial x_{2, \alpha_2}} + \frac{\partial I_{\alpha_1 \alpha_2 [3]}}{\partial x_{3, \alpha_3}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{1, \sigma_1} \partial x_{1, \sigma_1}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{2, \sigma_2} \partial x_{2, \sigma_2}} - \nu \frac{\partial^2 H_{\alpha_1 \alpha_2 \alpha_3}}{\partial x_{3, \sigma_3} \partial x_{3, \sigma_3}} = 0. \quad (\text{C.29})$$

As before in the transverse field representation (C.17)-(C.22), the above equations are coupled through the definition

$$\hat{H}_{\alpha_1 \dots \alpha_n \sigma_k} = \lim_{\mathbf{x}_{n+1} \rightarrow \mathbf{x}_k; \alpha_{n+1} \rightarrow \sigma_k} H_{\alpha_1 \dots \alpha_n \alpha_{n+1}}, \quad \forall k \text{ between } 1 \leq k \leq n, \quad (\text{C.30})$$

and are accompanied by the incompressibility constraints

$$\frac{\partial H_{\alpha_1 \dots \alpha_n}}{\partial x_{k, \alpha_k}} = 0, \quad \forall k \text{ between } 1 \leq k \leq n, \quad (\text{C.31})$$

$$\frac{\partial I_{\alpha_1 \dots \alpha_{l-1} \alpha_{l+1} \dots \alpha_n [l]}}{\partial x_{k, \alpha_k}} = 0, \quad \forall k, l \text{ between } 1 \leq (k, l) \leq n, \text{ for } k \neq l. \quad (\text{C.32})$$

Note that the above system agrees with the MPC hierarchy of equations derived in Oberlack & Rostek (2010), from which also the above notation has been borrowed (see also Frewer *et al.* (2014a,b)). These are i) the instantaneous (equal-time) multi-point velocity correlation functions of order  $n \geq 1$

$$H_{\alpha_{\{n\}}} = H_{\alpha_1 \dots \alpha_n}(\mathbf{x}_1, \dots, \mathbf{x}_n, t) = \langle u_{\alpha_1}(\mathbf{x}_1, t) \dots u_{\alpha_n}(\mathbf{x}_n, t) \rangle, \quad (\text{C.33})$$

where  $\hat{H}_{\alpha_1 \dots \alpha_n \sigma_k}(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_k, t)$  is a lower dimensional moment which emerges from the next higher one  $H_{\alpha_1 \dots \alpha_n \alpha_{n+1}}(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}, t)$  in the limit of zero correlation length  $|\mathbf{x}_{n+1} - \mathbf{x}_k| \rightarrow 0$  along with a corresponding change in the tensor index  $\alpha_{n+1} \rightarrow \sigma_k$ , for all  $k$  within the range  $1 \leq k \leq n$  (C.30), and then ii) the instantaneous (equal-time) multi-point pressure velocity correlation functions of order  $(n - 1)$

$$I_{\alpha_{\{n-1\}}} [l] = \langle u_{\alpha_1}(\mathbf{x}_1, t) \dots u_{\alpha_{l-1}}(\mathbf{x}_{l-1}, t) \cdot p(\mathbf{x}_l, t) \cdot u_{\alpha_{l+1}}(\mathbf{x}_{l+1}, t) \dots u_{\alpha_n}(\mathbf{x}_n, t) \rangle, \quad (\text{C.34})$$

where  $p$  is the instantaneous pressure field, which in the unbounded domain evolves according to Poisson's equation as (see e.g. McComb (1990))

$$p(\mathbf{x}, t) = \int d^3 \mathbf{x}' \frac{\partial'^2 u_{\alpha}(\mathbf{x}', t) u_{\beta}(\mathbf{x}', t)}{4\pi |\mathbf{x} - \mathbf{x}'|}. \quad (\text{C.35})$$



## D. Derivation of the ODE moment equations from the underlying PDE

In a similar fashion as the infinite (lower level) Friedmann-Keller hierarchy of moments (2.13) emerge from the underlying (higher level) Hopf equation (2.6), the analogous lower level ODE moment equations (2.19) are induced by the higher level PDE (2.18) in the following way: Multiplying the unbounded PDE for  $u = u(x, t)$

$$\partial_t u = \partial_x^2 u - \lambda \cdot x^2 u, \quad (\text{D.1})$$

by  $x^n$ , for all  $n \geq 0$ , and integrating over  $\mathbb{R}$  by assuming that partial integration is justified, i.e. by assuming the natural boundary conditions

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \partial_x u(x, t) = 0, \quad \forall t \geq 0, \quad (\text{D.2})$$

then one obtains for each order  $n$  the following infinite system of coupled ODEs

$$\frac{du_n}{dt} = n \cdot (n - 1) \cdot u_{n-2} - \lambda \cdot u_{n+2}, \quad \forall n \geq 0, \quad (\text{D.3})$$

where the moments  $u_n = u_n(t)$  are defined by

$$u_n(t) = \int_{-\infty}^{\infty} x^n \cdot u(x, t) dx, \quad \forall n \geq 0. \quad (\text{D.4})$$

Formulating (D.1) as a Cauchy problem, by including the initial condition

$$u(x, 0) = \phi(x), \quad \text{with} \quad \int_{-\infty}^{\infty} \phi(x) dx = 1, \quad (\text{D.5})$$

where  $\phi$  is some arbitrary but integrable function, it will correspondingly restrict the infinite system of equations (D.3) to a first order ODE initial value problem, in that it has to satisfy the restrictions

$$u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) dx, \quad \forall n \geq 0, \quad \text{with} \quad u_0(0) = 1. \quad (\text{D.6})$$

## References

- ANDREEV, V. K., KAPTSOV, O. V., PUKHNACHOV, V. V. & RODIONOV, A. A. 1998 *Applications of Group-Theoretical Methods in Hydrodynamics*. Kluwer Academic Press.
- AVSARKISOV, V., OBERLACK, M. & HOYAS, S. 2014 New scaling laws for turbulent Poiseuille flow with wall transpiration. *J. Fluid Mech.* **746**, 99–122.
- BILA, N. 2011 On a new method for finding generalized equivalence transformations for differential equations involving arbitrary functions. *J. Sym. Comp.* **46**, 659–671.
- BLUMAN, G. W. & KUMEI, S. 1996 *Symmetries and Differential Equations*, 2nd edn. Springer Verlag.
- DYSON, F. 2004 A meeting with Enrico Fermi: How one intuitive physicist rescued a team from fruitless research. *Nature* **427** (6972), 297.
- FREWER, M., KHUJADZE, G. & FOYSI, H. 2014a On the physical inconsistency of a new statistical scaling symmetry in incompressible Navier-Stokes turbulence. [arXiv:1412.3061](https://arxiv.org/abs/1412.3061).

- FREWER, M., KHUJADZE, G. & FOYSI, H. 2014*b* A critical examination of the statistical symmetries admitted by the Lundgren-Monin-Novikov hierarchy of unconfined turbulence. [\*arXiv:1412.6949\*](#).
- FRISCH, U. 1995 *Turbulence. The Legacy of A.N. Kolmogorov*. Cambridge University Press.
- FUSHCHICH, W. I., SHTELEN, W. M. & SEROV, N. I. 1993 *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*. Springer Verlag.
- HOPF, E. 1952 Statistical hydromechanics and functional calculus. *J. Rational Mech. Anal.* **1**, 87–123.
- HOSOKAWA, I. 2006 Monin-Lundgren hierarchy versus the Hopf equation in the statistical theory of turbulence. *Phys. Rev. E* **73**, 067301.
- HYDON, P. E. 2000 *Symmetry Methods for Differential Equations: A Beginner's Guide*. Cambridge University Press.
- IBRAGIMOV, N. H. 1994 *Lie Group Analysis of Differential Equations, CRC Handbook*, vol. I-III. CRC Press.
- IBRAGIMOV, N. H. 2004 Equivalence groups and invariants of linear and non-linear equations. *Archives of ALGA* **1**, 9–69.
- KELLER, L. V. & FRIEDMANN, A. A. 1924 Differentialgleichung für die turbulente Bewegung einer kompressiblen Flüssigkeit. *Proc. 1st Intern. Congr. Appl. Mech.* pp. 395–405.
- MCCOMB, W. D. 1990 *The Physics of Fluid Turbulence*. Clarendon Press Oxford.
- MELESHKO, S. V. 1996 Generalization of the equivalence transformations. *J. Nonlin. Math. Phys.* **3** (1-2), 170–174.
- MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics, Vol. I-II*. MIT Press.
- OBERLACK, M. & ROSTECK, A. 2010 New statistical symmetries of the multi-point equations and its importance for turbulent scaling laws. *Discrete Continuous Dyn. Syst. Ser. S* **3**, 451–471.
- OBERLACK, M. & ROSTECK, A. 2011*a* Applications of the new symmetries of the multi-point correlation equations. *Journal of Physics: Conference Series* **318** (4), 042011.
- OBERLACK, M. & ROSTECK, A. 2011*b* Turbulent scaling laws and what we can learn from the multi-point correlation equations. In *Proceedings of the 7th International Symposium on Turbulent and Shear Flow Phenomena (TSFP-7)*.
- OBERLACK, M., ROSTECK, A. & AVSARKISOV, V. 2014 Can we obtain first principle results for turbulence statistics? *International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics*, HEFAT2014.
- OBERLACK, M. & ZIELENIEWICZ, A. 2013 Statistical symmetries and its impact on new decay modes and integral invariants of decaying turbulence. *Journal of Turbulence* **14** (2), 4–22.
- OLVER, P. J. 1993 *Applications of Lie Groups to Differential Equations*, 2nd edn. Springer Verlag.
- OVSIIANNIKOV, L. V. 1982 *Group Analysis of Differential Equations*, 2nd edn. Academic Press.

- POLYANIN, A. D. 2002 *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. CRC Press.
- STEPHANI, H. 1989 *Differential Equations. Their solutions using symmetries*. Cambridge University Press.
- STEWART, A. M. 2008 Longitudinal and transverse components of a vector field. *arXiv:0801.0335*.
- WACŁAWCZYK, M., STAFFOLANI, N., OBERLACK, M., ROSTECK, A., WILCZEK, M. & FRIEDRICH, R. 2014 Statistical symmetries of the Lundgren-Monin-Novikov hierarchy. *Phys. Rev. E* **90** (1), 013022.